

15 Classical Mechanics

15.0. Introduction

The development of any branch of Physics is based on certain laws which, in turn, include certain concepts of fundamental nature. The development of *Classical Mechanics* is based on well known Newton's three laws (i) the law of inertia or Galileo law (ii) the law of force (iii) the law of action and reaction. These laws include the concepts of absolute mass, absolute space and absolute time. The classical mechanics explains correctly the motion of celestial bodies like planets, stars and macroscopic as well as microscopic terrestrial bodies moving with non-relativistic speeds (i.e. $v \ll c$; c being the speed of light). The other class of mechanical phenomena such as the non-relativistic motion of atoms, molecules, electrons, protons etc., is explained more correctly by Quantum Mechanics and the relativistic motion by Relativistic Quantum Mechanics. In this chapter we shall discuss the development of *Classical Mechanics*.

15.1. Review of Mechanics for a System of Particles

(i) *Theorem of Conservation of Linear Momentum.* The linear momentum of a particle of mass m and velocity vector $\mathbf{v}$ is defined as $\mathbf{p} = m\mathbf{v}$.

Therefore the net linear momentum for a system of n -particles is

$$\mathbf{P} = \sum_{i=1}^n \mathbf{p}_i = \sum_{i=1}^n m_i \mathbf{v}_i$$

From Newton's second and third laws, we obtain

$$\mathbf{F}^{\text{ext}} = \frac{d\mathbf{P}}{dt}$$

i.e. the rate of change of linear momentum of a system of particles is equal to the net external force acting on the system.

If $\mathbf{F}^{\text{ext}} = 0$, $\frac{d\mathbf{P}}{dt} = 0$ and integrating we get $\mathbf{P} = \text{constant}$. Thus, if the *net external force acting on a system of particles is zero, the net linear momentum of the system remains constant*. This is the principle of conservation of linear momentum of a system of particles.

2. Angular Momentum. The angular momentum of a particle of linear momentum $\mathbf{p}$ ($= m\mathbf{v}$) and having position vector $\mathbf{r}$ relative to an arbitrary origin is defined as

$$\mathbf{J} = \mathbf{r} \times \mathbf{p}$$

For a system of n particles; we have

$$\mathbf{J} = \sum_i \mathbf{J}_i = \sum_i \mathbf{r}_i \times \mathbf{p}_i$$

where i denotes i th particle.

Differentiating above equation and noting that

$$\frac{d\mathbf{r}_i}{dt} \times \mathbf{p}_i = \mathbf{v}_i \times \mathbf{p}_i = 0; \text{ we get}$$

$$\frac{d\mathbf{J}}{dt} = \sum_i \mathbf{r}_i \times \frac{d\mathbf{p}_i}{dt} = \sum_i \mathbf{r}_i \times \mathbf{F}_i$$

where $\mathbf{F}_i = \frac{d\mathbf{p}_i}{dt}$ = net force acting on i th particle.

As internal forces occur in pairs of equal and opposite forces; hence the net internal force acting on the system of particles as a whole is zero. Thus,

$$\frac{d\mathbf{J}}{dt} = \sum_i \mathbf{r}_i \times \mathbf{F}_i^{\text{ext}} = \boldsymbol{\tau}^{\text{ext}}$$

where $\boldsymbol{\tau}^{\text{ext}} = \sum_i \mathbf{r}_i \times \mathbf{F}_i^{\text{ext}}$ is the torque arising due to external force only.

If $\boldsymbol{\tau}^{\text{ext}} = 0$, $\frac{d\mathbf{J}}{dt} = 0$ or $\mathbf{J} = \text{constant}$.

Thus, *if external torque acting on a system of particles is zero, the angular momentum of the system remains constant.* This is conservation theorem of angular momentum.

3. Conservation of Energy. If the work done by a force is independent of path, the force is said to be conservative.

If the forces acting on the particles are conservative : its mechanical energy (kinetic energy plus potential energy) remains constant.

On the other hand if the forces are non-conservative the total energy of universe (mechanical energy plus chemical energy + sound energy + light energy + heat energy etc.) remains constant.

15.2. Basic Concepts

15.2(a) Constraints. In practice the motion of a particle or system of particles is generally restricted in some way. Some of the examples of restricted motion are :

- (1) The motion of rigid bodies is always such that the distance between any two particles remains unchanged.
- (2) The motion of point mass of a simple pendulum is restricted since the point mass always remains at a constant distance from the point of suspension.
- (3) The motion of the gas molecules within a container is restricted by the walls of the vessel since gas molecules can move only inside the container.
- (4) The motion of a particle placed on the surface of a solid sphere is restricted so that it can only move either on the surface or outside the sphere.

From the above examples we see that generally the motion of a particle or system of particles is restricted or limited. *The limitations (or geometrical restrictions) on the motion of a particle or system of particles are generally known as constraints.*

Classification of constraints. The constraints may be classified in the ways given below :

(i) **Holonomic and Non-holonomic Constraints.** Let $\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_n$ be the position coordinates of a system and let t denote the time : then if conditions of all the constraints are expressed as equations having the form

$$f(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \dots, \mathbf{r}_n, t) = 0 \quad \dots (1)$$

then the constraints are said to be *holonomic*. If the conditions of the constraints are not so expressed, they are called *non-holonomic constraints*.

Some of the examples of holonomic constraints are as follows :

(1) The constraints involved in the motion of rigid bodies in which the distance between any two particular points is always fixed, are *holonomic* since the conditions of constraints are expressed as

$$(\mathbf{r}_i - \mathbf{r}_j)^2 - c_{ij}^2 = 0$$

(2) The constraints involved in the motion of the point mass of a simple pendulum are *holonomic*. In this case the point mass remains at a constant distance l from the point of suspension whose position vector is $\mathbf{a}$ and hence the conditions of constraints are expressed as

$$(\mathbf{r} - \mathbf{a})^2 = l^2,$$

where $\mathbf{r}$ is the position vector of the point mass.

(3) The constraints involved when a particle is restricted to move along a curve of surface are *holonomic*.

Some of the examples of *non-holonomic constraints* are :

(1) The constraints involved in the motion of the particle placed on the surface of solid sphere are non-holonomic. The conditions of constraints in this case are expressed as

$$r^2 - a^2 \geq 0,$$

where a is the radius of sphere. This is an inequality and hence not in the form of equation (1).

(2) The constraints involved in the motion of the molecules in a gas container are non-holonomic.

(3) An object rolling on a rough surface without slipping is also an example of non-holonomic constraints.

There are also other examples of non-holonomic constraints which cannot be expressed in mathematical form.

(ii) **Scleronomic and Rheonomic Constraints.** If the constraints are independent of time, they are termed as *scleronomic* ; but they contain time explicitly, they are called *rheonomic*.

A bead sliding on a moving wire is an example of rheonomic constraints.

In the solution of mechanical problems the constraints introduce two types of difficulties :

(1) The co-ordinates r_i are connected by the equations of constraints, therefore they are not independent

(2) In general, the forces are required to maintain the constraints in the system. These forces are not known initially, i.e., the forces present in the problem are not specified directly.

In the holonomic constraints the first difficulty is solved by the use of *generalised co-ordinates* and the second difficulty is removed by eliminating the forces from equations of motion at an early stage by formulating the mechanics properly.

15.2 (b) Generalised Co-ordinates

In the solution of most of the mechanical problems it is more convenient to use some other set of co-ordinates instead of cartesian co-ordinates. For example, in the case of a particle moving on the surface of a sphere, the correct co-ordinates are spherical co-ordinates r, θ, ϕ where θ and ϕ are only two variable quantities.

Let there be a particle or system of n particles moving under possible constraints, for example, a point mass of the simple pendulum or rigid body moving along an inclined plane. Then there will be a minimum number of independent co-ordinates required to specify the motion of a particle or system of particles. The independent co-ordinates sufficient in number to specify unambiguously the system configuration are called *generalised co-ordinates* and are denoted by $q_1, q_2, q_3, \dots, q_k, \dots, q_f$ where f is the total number of generalised co-ordinates or degrees of freedom.

In the case of a system of n particles there are $3n$ independent cartesian co-ordinates; but if there exist holonomic constraints expressed by l equations of the form of equation (1) then these l equations may be used to eliminate l of the $3n$ co-ordinates and now there are $3n - l = f$ (say) independent co-ordinates or degrees of freedom. In this case there are $3n - l = f$ generalised co-ordinates.

In order to choose a suitable set of generalised co-ordinates in a given problem, we must take into account the following three principles :

- (i) They specify the configuration of the system.
- (ii) They must be varied arbitrarily and independently of each other, without violating the constraints of the system.
- (iii) They must simplify mathematical calculations without affecting informative and enlightening solution.

The number of co-ordinates satisfying conditions (1) and (2) is the number of degrees of freedom. Condition (3) then implies that after ascertaining about the number of degrees of freedom, we choose a suitable co-ordinate system to achieve mathematical simplicity in setting up equations of motion as well as in their solution.

Illustrations for Choosing Generalised Coordinate

1. Let there be a particle of mass m constrained to move on a circular wire in the X - Y plane having co-ordinates (x, y) at any instant. Let r be the radius of the circle; then we have

$$x = r \cos \theta$$

$$\text{and } y = r \sin \theta$$

Thus the motion of the particle can be satisfied by the use of generalised co-ordinate θ alone, since r is constant.

Unlike Cartesian co-ordinates generalised co-ordinates are not divided into three convenient groups of co-ordinates, which may form vectors.

2. The motion of particle constrained to move on the surface of a sphere of radius ' a ' with origin at centre of sphere may be described by Cartesian coordinates x, y, z . These coordinates are related as

$$x^2 + y^2 + z^2 = a^2$$

Clearly x, y, z are related and hence all three coordinates are not independent but only two will be independent. Let these coordinates be $q_1 = \frac{x}{a}$ and $q_2 = \frac{y}{a}$, then

$$q_3 = \sqrt{\frac{a^2 - x^2 - y^2}{a^2}} = (1 - q_1^2 - q_2^2)$$

Clearly, q_3 is not independent; so number of generalised coordinates will be 2.

But the cartesian coordinate so chosen are not convenient due to spherical symmetry of problem and hence the spherical polar coordinates will be most convenient. In spherical coordinates $r = a = \text{constant}$, so the generalised coordinates will be $q_1 = \theta$ and $q_2 = \phi$. These are related to Cartesian coordinates by

$$q_1 = \theta = \cos^{-1} \frac{z}{a}$$

$$q_2 = \phi = \tan^{-1} \frac{y}{x}$$

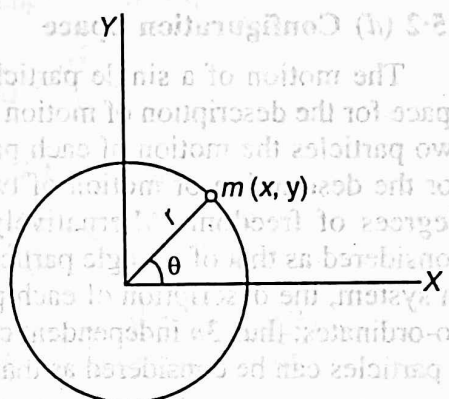


Fig. 15.1

15.2. (c) Transformation equations

The old co-ordinates can be represented as the functions of generalised co-ordinates. If x_i, y_i and z_i are the cartesian co-ordinates of i th particle of the system, then these cartesian co-ordinates can be expressed as functions of generalised co-ordinates $q_1, q_2, q_3, \dots, q_f$, i.e.

$$\left. \begin{aligned} x_i &= x_i(q_1, q_2, \dots, q_f, t) \\ y_i &= y_i(q_1, q_2, \dots, q_f, t) \\ z_i &= z_i(q_1, q_2, \dots, q_f, t) \end{aligned} \right\} \dots (2)$$

where t denotes the time.

If $\mathbf{r}_i$ is the position vector of i th particle, i.e. $\mathbf{r}_i = \hat{\mathbf{i}}x_i + \hat{\mathbf{j}}y_i + \hat{\mathbf{k}}z_i$, when $\hat{\mathbf{i}}, \hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ are unit vectors along x, y and z -axes respectively, then

$$\mathbf{r}_i = \mathbf{r}_i(q_1, q_2, q_3, \dots, q_f, t), \dots (3)$$

which is the vector form of equation (2).

The equations like (2) and (3) are called *transformation equations*. The functions and their derivatives in the above two equations are supposed to be continuous. The equations also contain the constraints explicitly.

15.2 (d) Configuration Space

The motion of a single particle can be described by a set of three co-ordinates, i.e., the space for the description of motion of a single particle must be three dimensional. In the case of two particles the motion of each particle will be described by a set of three co-ordinates, i.e., for the description of motion of two particles, we require six independent coordinates or six degrees of freedom. Alternatively, we may say that the problem of two particles can be considered as that of a single particle in six dimensional space. Similarly, if there are n particles in system, the description of each particle of the system will require a set of three independent co-ordinates; thus $3n$ independent co-ordinates (or degrees of freedom) in total i.e., a problem of n particles can be considered as that of a single particle in $3n$ dimensional space.

If the system is such that the conditions of constraints are expressed by l equations, then there will be $(3n - l) = f$ independent co-ordinates or degree of freedom. In this case the problem of n particles can be considered as that of a single particle in f dimensional space.

Thus for a system of n -particles $3n$ dimensional space in the absence of constraints and $\{(3n - l)\}$ dimensional space in the presence of constraints is called the *configuration space* where the conditions of constraints are expressed by l equations.

15.2 (e). Generalised Notations

Now we shall express displacement, velocity, acceleration, momentum, force and potential energy in terms of generalised coordinates.

1. Generalised Displacement. Consider a system of n -particles having position vectors $\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_i, \dots$ in cartesian coordinates at time t . At any instant the displacement of i th particle is $\delta \mathbf{r}_i$. If $q_1, q_2, \dots, q_f$ are the generalised coordinates of the system, then the transformation equation is

$$\mathbf{r}_i = \mathbf{r}_i(q_1, q_2, \dots, q_f, t) \dots (1)$$

$$\delta \mathbf{r}_i = \sum_{k=1}^f \frac{\partial \mathbf{r}_i}{\partial q_k} \delta q_k + \frac{\partial \mathbf{r}_i}{\partial t} \delta t$$

If we assume time to be fixed $\delta t = 0$, we have

$$\therefore \delta \mathbf{r}_i = \sum_{k=1}^f \frac{\partial \mathbf{r}_i}{\partial q_k} \delta q_k \dots (2)$$

The variations in generalised coordinates q_k 's are called **generalised displacements**.

2. Generalised Velocity. By transformation equation

$$\mathbf{r}_i = \mathbf{r}_i(q_1, q_2, \dots, q_f, t)$$

The time rate of change of displacement of position vector $\mathbf{r}_i$ is called the velocity of i th particle.

We have

$$\begin{aligned} \frac{d\mathbf{r}_i}{dt} &= \sum_k \frac{\partial \mathbf{r}_i}{\partial q_k} \frac{dq_k}{dt} + \frac{\partial \mathbf{r}_i}{\partial t} \\ \Rightarrow \dot{\mathbf{r}}_i &= \sum_k \frac{\partial \mathbf{r}_i}{\partial q_k} \dot{q}_k + \frac{\partial \mathbf{r}_i}{\partial t} \end{aligned} \quad \dots(3)$$

In this expression the terms $\dot{q}_k$ ($k = 1, 2, \dots$) represents the generalised velocity.

3. Generalised Acceleration. Differentiating (3) with respect to time t , we get

$$\begin{aligned} \ddot{\mathbf{r}}_i &= \frac{d}{dt}(\dot{\mathbf{r}}_i) = \frac{d}{dt} \left(\sum_k \frac{\partial \mathbf{r}_i}{\partial q_k} \dot{q}_k + \frac{\partial \mathbf{r}_i}{\partial t} \right) \\ &= \sum_k \frac{\partial \mathbf{r}_i}{\partial t} \ddot{q}_k + \sum_k \frac{d}{dt} \left(\frac{\partial \mathbf{r}_i}{\partial q_k} \right) \dot{q}_k + \frac{d}{dt} \left(\frac{\partial \mathbf{r}_i}{\partial t} \right) \\ &= \sum_k \frac{\partial \mathbf{r}_i}{\partial t} \ddot{q}_k + \sum_k \frac{\partial \dot{\mathbf{r}}_i}{\partial q_k} \dot{q}_k + \frac{\partial \dot{\mathbf{r}}_i}{\partial t} \end{aligned}$$

Substituting value of $\dot{\mathbf{r}}_i$ from (3), and changing index from k to l , we get

$$\begin{aligned} \ddot{\mathbf{r}}_i &= \sum_k \frac{\partial \mathbf{r}_i}{\partial t} \ddot{q}_k + \sum_k \frac{\partial}{\partial q_k} \left(\sum_l \frac{\partial \mathbf{r}_i}{\partial q_l} \dot{q}_l + \frac{\partial \mathbf{r}_i}{\partial t} \right) \dot{q}_k + \frac{\partial}{\partial t} \left(\sum_l \frac{\partial \mathbf{r}_i}{\partial q_l} \dot{q}_l + \frac{\partial \mathbf{r}_i}{\partial t} \right) \\ &= \sum_k \frac{\partial \mathbf{r}_i}{\partial t} \ddot{q}_k + \sum_k \sum_l \frac{\partial^2 \mathbf{r}_i}{\partial q_k \partial q_l} \dot{q}_l \dot{q}_k + \sum_k \frac{\partial^2 \mathbf{r}_i}{\partial q_k \partial t} \dot{q}_k + \sum_k \frac{\partial^2 \mathbf{r}_i}{\partial t \partial q_l} \dot{q}_l + \frac{\partial^2 \mathbf{r}_i}{\partial t^2} \\ &= \sum_k \frac{\partial \mathbf{r}_i}{\partial t} \ddot{q}_k + \sum_k \sum_l \frac{\partial^2 \mathbf{r}_i}{\partial q_k \partial q_l} \dot{q}_l \dot{q}_k + 2 \sum_k \frac{\partial^2 \mathbf{r}_i}{\partial q_k \partial t} \dot{q}_k + \frac{\partial^2 \mathbf{r}_i}{\partial t^2} \end{aligned} \quad \dots(4)$$

The term $\ddot{q}_k$ is the generalised acceleration. Equation (4) shows that Cartesian components of acceleration are not the linear functions of generalised acceleration $\ddot{q}_k$ alone, but depend quadratically and linearly on generalised velocity components $\dot{q}_k$ as well. However, according to Lagrange's approach the computation of second derivatives of generalised coordinates is not essential.

4. Generalised Momentum. The kinetic energy of a system of particles in Cartesian coordinates is given by

$$T = \sum_i \frac{1}{2} m_i \dot{\mathbf{r}}_i^2 = \sum_i \frac{1}{2} m_i (\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2) \quad \dots(5)$$

where $\dot{x}_i$, $\dot{y}_i$ and $\dot{z}_i$ are components of velocity of i th particle along x , y and z -directions respectively.

The momentum of i th particle along x -direction is

$$p_{ix} = m_i \dot{x}_i \quad \dots(6)$$

Differentiating equation (5) with respect to $\dot{x}_i$, we get

$$\frac{\partial T}{\partial \dot{x}_i} = m_i \dot{x}_i$$

In view of equation (6), we have

$$p_{ix} = \frac{\partial T}{\partial \dot{x}_i} = m_i \dot{x}_i$$

In analogy with this equation, the generalised momentum associated with generalised coordinate q_k is expressed as

$$p_k = \frac{\partial T}{\partial \dot{q}_k} = m_k \dot{q}_k \quad \dots(7)$$

5. Generalised Force. If $\vec{F}_i$ is force acting on i th particle, $\delta \vec{r}_i$ is an arbitrary small displacement; then work done by force $\vec{F}_i$ during this displacement is given by

$$dW_i = \vec{F}_i \cdot \delta \vec{r}_i$$

Total work done by the forces

$$W = \sum_i \vec{F}_i \cdot \delta \vec{r}_i$$

As

$$\vec{r}_i = \vec{r}_i(q_1, q_2, \dots, t)$$

$$\delta \vec{r}_i = \sum_k \frac{\partial \vec{r}_i}{\partial q_k} \delta q_k$$

$\therefore$

$$W = \sum_i \sum_k \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_k} \delta q_k = \sum_k \left(\sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_k} \right) \delta q_k = \sum_k Q_k \delta q_k \quad \dots(9)$$

where

$$Q_k = \sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_k} \quad \dots(10)$$

is called the *generalised force*, associated with coordinate q_k . It may be noted that the expressions for work done in Cartesian coordinates $W = \sum_i \vec{F}_i \cdot \delta \vec{r}_i$ and in generalised coordinates

$\sum_k Q_k \delta q_k$ are analogous. It may be noted that it is not necessary that the dimensions of Q_k are that of force; but it is certain that the product of generalised force and generalised displacement has the dimensions of work.

15.2 (f) Principle of Virtual Work

Imagine two possible configurations of a system of particles at any instant of time which are consistent with the forces and constraints. If the system of particles goes from one configuration to the other, each particle of the system is imagined to be displaced by an infinitesimal vector $\delta \vec{r}_i$ and is called the *virtual displacement* to distinguish from the *actual displacement* $d\vec{r}_i$ which occurs in a time interval dt during which the forces and constraints may be changing.

In order for a system to be in equilibrium, the resultant force acting on each particle of the system must be zero, i.e., $\vec{F}_i = 0$, where $\vec{F}_i$ is the force acting on i th particle of the system. then clearly, the dot product of force $\vec{F}_i$ and the virtual displacement $\delta \vec{r}_i$ due to force $\vec{F}_i$, which is the virtual work must vanish.

$$i.e., \quad \vec{F}_i \cdot \delta \vec{r}_i = \text{virtual work} = 0.$$

Summing over all the particles of the system, we have

$$\sum_i \vec{F}_i \cdot \delta \vec{r}_i = 0 \quad \dots(4)$$

since the sum of the vanishing products over all the particles of the system is also zero.

If the constraints are present, then the force $\mathbf{F}_i$ acting on the i th particle is written as the sum of constraint force and actual force, i.e.,

$$\mathbf{F}_i = \mathbf{F}_i^a + \mathbf{f}_i \quad \dots(5)$$

$\mathbf{F}_i^a$ being the actual force and $\mathbf{f}_i$ being the force of constraint acting on i th particle.

Putting value of $\mathbf{F}_i$ from equation (5) in equation (4), we have

$$\sum_i (\mathbf{F}_i^a + \mathbf{f}_i) \cdot \delta \mathbf{r}_i = 0$$

or

$$\sum_i (\mathbf{F}_i^a + \mathbf{f}_i) \cdot \delta \mathbf{r}_i = 0 \quad \dots(6)$$

Now restricting ourselves to the system where the virtual work done by the forces of constraints, is zero i.e.,

$$\sum_i \mathbf{f}_i \cdot \delta \mathbf{r}_i = 0.$$

This statement holds for a large number of systems consisting of constraints. For example, consider a particle constrained to move on a surface. In this case, the force of constraint is perpendicular to the surface while the virtual displacement is along the tangent to the surface. Therefore the virtual work done by the force of constraint is zero, i.e., $\mathbf{f}_i \cdot \delta \mathbf{r}_i = 0$. The statement does not hold for the systems consisting of frictional forces. Therefore we should not consider the system consisting of frictional forces.

Now equation (6) reduces to

$$\sum_i \mathbf{F}_i^a \cdot \delta \mathbf{r}_i = 0. \quad \dots(7)$$

This is the condition for equilibrium of a system according to which "A system of particles is in equilibrium only if the total virtual work of the actual or applied forces is zero." This is known as the principle of virtual work.

15.2 (g) D' Alembert's Principle

The principle of virtual work is applicable to the statics of a system of particles : but it can be related to obtain a similar principle of dynamics.

According to Newton's second law of motion, the force is defined as rate change of momentum, i.e.,

$$\mathbf{F}_i = \frac{d\mathbf{p}_i}{dt} = \dot{\mathbf{p}}_i$$

where $\mathbf{p}_i$ is the momentum of i th particle due to force $\mathbf{F}_i$ acting on i th particle

$$\therefore \mathbf{F}_i - \dot{\mathbf{p}}_i = 0 \quad \dots(8)$$

According to the above equation a moving system of particles can be considered to be in equilibrium under the force $\mathbf{F}_i - \dot{\mathbf{p}}_i$ i.e., the actual applied force $\mathbf{F}_i$ plus an additional force $-\dot{\mathbf{p}}_i$ which is known as reversed effective force on i th particle.

Now replacing $\mathbf{F}_i$ of equation (4) by $(\mathbf{F}_i - \dot{\mathbf{p}}_i)$, we have

$$\sum_i (\mathbf{F}_i - \dot{\mathbf{p}}_i) \cdot \delta \mathbf{r}_i = 0 \quad \dots(9)$$

If the forces of constraints are present, from equation (5) we have

$$\mathbf{F}_i = \mathbf{F}_i^a + \mathbf{f}_i, \quad \dots(10)$$

Putting value of $\mathbf{F}_i$ from equation (10) in equation (9), we get

$$\sum_i \{(\mathbf{F}_i^a + \mathbf{f}_i) - \dot{\mathbf{p}}_i\} \cdot \delta \mathbf{r}_i = 0$$

$$\sum_i \{(\mathbf{F}_i^a - \dot{\mathbf{p}}_i) \cdot \delta \mathbf{r}_i + \sum_i \mathbf{f}_i \cdot \delta \mathbf{r}_i = 0. \quad \dots(11)$$

Again, restricting ourselves to the cases where the virtual work done by the forces of constraints vanishes, i.e.,

$$\sum_i \mathbf{f}_i \cdot \delta \mathbf{r}_i = 0.$$

Equation (11) reduces to

$$\text{or} \quad \sum_i (\mathbf{F}_i^a - \dot{\mathbf{p}}_i) \cdot \delta \mathbf{r}_i = 0 \quad \dots(12)$$

which is called *D' Alembert's principle*.

It is to be noted here that we have restricted ourselves to the systems where the virtual work done by the forces of constraints disappears. With this in mind we can drop the superscript *a* in equation (5), i.e., D' Alembert's principle may be written as

$$\boxed{\sum_i (\mathbf{F}_i - \dot{\mathbf{p}}_i) \cdot \delta \mathbf{r}_i = 0.} \quad \dots(13)$$

Thus D' Alembert's principle states that the virtual work done by effective force (external force + reversed effective force) on a dynamical system is zero.

15.3 The Lagrangian Formulation

15.3. (a) Lagrange's Equations from D' Alembert's Principle

Let there be a system of particles which requires *f* independent generalised co-ordinates (or degrees of freedom) to specify the states of its particles.

According to transformation equations the position vectors of the particles $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \dots, \mathbf{r}_i, \dots$ are expressed as the functions of generalised co-ordinates $q_1, q_2, q_3, \dots, q_k, \dots, q_f$ and the time *t*, i.e.,

$$\mathbf{r}_1 = \mathbf{r}_1(q_1, q_2, q_3, \dots, q_k, \dots, q_f, t),$$

$$\mathbf{r}_2 = \mathbf{r}_2(q_1, q_2, q_3, \dots, q_k, \dots, q_f, t),$$

$$\dots \dots \dots \dots \dots \dots \dots$$

$$\text{In general,} \quad \mathbf{r}_i = \mathbf{r}_i(q_1, q_2, q_3, \dots, q_k, \dots, q_f, t) \quad \dots(1)$$

Consider any particle of the system (*i*th particle) of mass *m_i* and acted upon by an external force $\mathbf{F}_i$; then according to D' Alembert's principle, we have

$$\sum_i (\mathbf{F}_i - \dot{\mathbf{p}}_i) \cdot \delta \mathbf{r}_i = 0. \quad \dots(2)$$

where $\dot{\mathbf{p}}_i$ is the inertial force for *i*th particle and $\delta \mathbf{r}_i$ is the virtual displacement of *i*th particle due to action of force $\mathbf{F}_i$.

Now since $\dot{\mathbf{p}}_i = m_i \ddot{\mathbf{r}}_i$, therefore equation (2) becomes

$$\sum_i (\mathbf{F}_i - m_i \ddot{\mathbf{r}}_i) \cdot \delta \mathbf{r}_i = 0. \quad \dots(3)$$

The virtual displacement $\delta \mathbf{r}_i$ in terms of generalised co-ordinates from equation (1) is given by

$$\delta \mathbf{r}_i = \sum_k \frac{\partial \mathbf{r}_i}{\partial q_k} \delta q_k \quad \dots(4)$$

The above equation does not involve the variation of time since the virtual displacements, by definition, do not consist of the variation of time, but consists of only the displacements of the co-ordinates.

Equation (3) can be written as

$$\sum_i m_i \ddot{\mathbf{r}}_i \cdot \delta \mathbf{r}_i = \sum_i \mathbf{F}_i \cdot \delta \mathbf{r}_i \quad \dots(5)$$

Putting the value of $\delta \mathbf{r}_i$ from equation (4) in (5), we get

$$\sum_i \sum_k m_i \ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} \delta q_k = \sum_i \sum_k \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} \delta q_k \quad \dots(6)$$

$$\text{Since } \frac{d}{dt} \left(\dot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} \right) = \ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} + \dot{\mathbf{r}}_i \cdot \frac{d}{dt} \left(\frac{\partial \mathbf{r}_i}{\partial q_k} \right),$$

$$\therefore \ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} = \frac{d}{dt} \left(\dot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} \right) - \dot{\mathbf{r}}_i \cdot \frac{d}{dt} \left(\frac{\partial \mathbf{r}_i}{\partial q_k} \right) \quad \dots(7)$$

Putting the value of $\ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k}$ from above equation in (6), we get

$$\sum_i \sum_k m_i \left[\frac{d}{dt} \left(\dot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} \right) - \dot{\mathbf{r}}_i \cdot \frac{d}{dt} \left(\frac{\partial \mathbf{r}_i}{\partial q_k} \right) \right] \delta q_k = \sum_i \sum_k \mathbf{F}_i \cdot \left(\frac{\partial \mathbf{r}_i}{\partial q_k} \right) \delta q_k \quad \dots(8)$$

Differentiating equation (1) partially with respect to t , we get

$$\begin{aligned} \dot{\mathbf{r}}_i &= \frac{\delta \mathbf{r}_i}{\delta t} = \frac{\partial \mathbf{r}_i}{\partial q_1} \dot{q}_1 + \frac{\partial \mathbf{r}_i}{\partial q_2} \dot{q}_2 + \dots + \frac{\partial \mathbf{r}_i}{\partial q_k} \dot{q}_k + \dots + \frac{\partial \mathbf{r}_i}{\partial q_f} \dot{q}_f + \frac{\partial \mathbf{r}_i}{\partial t} \\ &= \sum_{k=1}^f \frac{\partial \mathbf{r}_i}{\partial q_k} \dot{q}_k + \frac{\partial \mathbf{r}_i}{\partial t} \quad \dots(9) \end{aligned}$$

Differentiating equation (9) with respect to $\dot{q}_k$ partially, we get

$$\frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_k} = \frac{\partial \mathbf{r}_i}{\partial q_k} \quad \dots(10)$$

Again, differentiating equation (9) partially with respect to q_k , we get

$$\frac{\partial \dot{\mathbf{r}}_i}{\partial q_k} = \frac{\partial^2 \mathbf{r}_i}{\partial q_k \partial q_1} \dot{q}_1 + \frac{\partial^2 \mathbf{r}_i}{\partial q_k \partial q_2} \dot{q}_2 + \dots + \frac{\partial^2 \mathbf{r}_i}{\partial q_k \partial q_k} \dot{q}_k + \dots + \frac{\partial^2 \mathbf{r}_i}{\partial q_k \partial t} \quad \dots(11)$$

Also we have

$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial \mathbf{r}_i}{\partial q_k} \right) &= \frac{\partial}{\partial q_1} \left(\frac{\partial \mathbf{r}_i}{\partial q_k} \right) \dot{q}_1 + \frac{\partial}{\partial q_2} \left(\frac{\partial \mathbf{r}_i}{\partial q_k} \right) \dot{q}_2 + \dots + \frac{\partial}{\partial q_k} \left(\frac{\partial \mathbf{r}_i}{\partial q_k} \right) \dot{q}_k + \dots + \frac{\partial}{\partial q_f} \left(\frac{\partial \mathbf{r}_i}{\partial q_k} \right) \dot{q}_f + \frac{\partial}{\partial t} \left(\frac{\partial \mathbf{r}_i}{\partial q_k} \right) \\ &= \frac{\partial^2 \mathbf{r}_i}{\partial q_k \partial q_1} \dot{q}_1 + \frac{\partial^2 \mathbf{r}_i}{\partial q_k \partial q_2} \dot{q}_2 + \dots + \frac{\partial^2 \mathbf{r}_i}{\partial^2 q_k} \dot{q}_k + \dots + \frac{\partial^2 \mathbf{r}_i}{\partial t \partial q_k} \quad \dots(12) \end{aligned}$$

The order of differentiation may be changed if $\mathbf{r}_i$ is assumed to have second order partial derivatives. Therefore comparing equations (11) and (12).

$$\frac{d}{dt} \left(\frac{\partial \mathbf{r}_i}{\partial q_k} \right) = \frac{\partial \dot{\mathbf{r}}_i}{\partial q_k} \quad \dots(13)$$

$$\text{We have } \frac{d}{dt} \left(\dot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} \right) = \frac{d}{dt} \left(\dot{\mathbf{r}}_i \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_k} \right) \text{ from equation (10)}$$

$$= \frac{d}{dt} \frac{\partial}{\partial \dot{q}_k} \left(\frac{1}{2} \dot{\mathbf{r}}_i^2 \right) \quad \dots(14)$$

Making substitutions from equations (13) and (14), equation (8) takes the form

$$\sum_i \sum_k m_i \left[\frac{d}{dt} \frac{\partial}{\partial \dot{q}_k} \left(\frac{1}{2} \dot{\mathbf{r}}_i^2 \right) - \dot{\mathbf{r}}_i \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_k} \right] \delta q_k = \sum_i \sum_k \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} \delta q_k$$

or

$$\sum_i \sum_k \left[\frac{d}{dt} \frac{\partial}{\partial \dot{q}_k} \left(\frac{1}{2} m_i \dot{\mathbf{r}}_i^2 \right) - \frac{\partial}{\partial \dot{q}_k} \left(\frac{1}{2} m_i \dot{\mathbf{r}}_i^2 \right) \right] \delta q_k = \sum_i \sum_k \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} \delta q_k$$

or

$$\sum_k \left[\frac{d}{dt} \frac{\partial}{\partial \dot{q}_k} \left(\sum_i \frac{1}{2} m_i \dot{\mathbf{r}}_i^2 \right) - \frac{\partial}{\partial \dot{q}_k} \left(\sum_i \frac{1}{2} m_i \dot{\mathbf{r}}_i^2 \right) \right] \delta q_k = \sum_i \sum_k \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} \delta q_k \quad \dots (15)$$

But $\sum_i \frac{1}{2} m_i \dot{\mathbf{r}}_i^2 = T = \text{total kinetic energy of the system of particles} \quad \dots (16)$

and

$$\sum_i \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} = Q_k \quad \dots (17)$$

where Q_k 's are components of *generalised force*.

With these substitutions equation (15) becomes

$$\sum_k \left\{ \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_k} - \frac{\partial T}{\partial q_k} \right\} \delta q_k = \sum Q_k \delta q_k \quad \dots (18)$$

or

$$\sum_k \left\{ \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_k} - \frac{\partial T}{\partial q_k} - Q_k \right\} \delta q_k = 0. \quad \dots (19)$$

The displacements δq_k are independent due to independence of generalised co-ordinates q_k , therefore equation (19) holds only if the coefficients of δq_k vanish separately, i.e.,

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_k} - \frac{\partial T}{\partial q_k} - Q_k = 0$$

$$\therefore \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_k} - \frac{\partial T}{\partial q_k} = Q_k \quad \dots (19)$$

This is the general form of *Lagrange's equation*. There are f such equations corresponding to f generalised co-ordinates. The above equation has been derived for a system involving no constraints; but the equation also holds for the system involving *holonomic constraints* because in the case of holonomic systems the equations of constraints may be used to minimise the degrees of freedom and hence the generalised coordinates to specify the states of the particles of the system.

Cases 1. When the system is wholly conservative.

If the system is conservative, all the forces acting on the system can be derived from a potential function V ; therefore for i th particle, we have

$$\mathbf{F}_i = -\nabla V_i = -\frac{\partial V_i}{\partial \mathbf{r}_i}, \quad \dots (20)$$

when ∇ acts as directional differentiator.

From equations (17) and (20)

$$Q_k = \sum_i \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} = -\sum_i \frac{\partial V_i}{\partial \mathbf{r}_i} \cdot \frac{\partial \mathbf{r}_i}{\partial q_k}$$

$$= -\sum_i \frac{\partial V_i}{\partial q_k}$$

$$= -\frac{\partial}{\partial q_k} (\sum_i V_i) = -\frac{\partial V}{\partial q_k} \quad \dots (21)$$

where $V = \sum_i V_i =$ total potential energy of the system

Putting the value of Q_k from above equation in (19), we get

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_k} \right) - \frac{\partial T}{\partial q_k} = - \frac{\partial V}{\partial q_k}$$

or
$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_k} \right) - \frac{\partial T}{\partial q_k} + \frac{\partial V}{\partial q_k} = 0$$

or
$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_k} \right) - \frac{\partial}{\partial q_k} (T - V) = 0 \quad \dots (22)$$

The potential energy V is the function of position co-ordinates q_k and not of the *generalised velocities* $\dot{q}_k$. Therefore above equation may be written as

$$\frac{d}{dt} \frac{\partial}{\partial \dot{q}_k} (T - V) - \frac{\partial}{\partial q_k} (T - V) = 0. \quad \dots (23)$$

Substituting L for $T - V$, i.e., $L = T - V$, $\dots (24)$

where L is known as *Lagrangian function* or simply the *Lagrangian* and is defined as the difference between total kinetic and potential energies of the system."

With this substitution, equation (23) becomes

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} = 0. \quad \dots (25)$$

This is Lagrange's equation for a *conservative system*.

Case II. When the system is partly conservative and partly non-conservative.

Let the system be acted upon by non-conservative force F'_i along with conservative forces F_i . The work done by non-conservative forces F'_i in virtual displacements $\delta \mathbf{r}_i$ is given by

$$\begin{aligned} \delta W &= \sum_i \mathbf{F}'_i \cdot \delta \mathbf{r}_i \\ &= \sum \sum \mathbf{F}'_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} \delta q_k. \end{aligned} \quad \dots (26)$$

[from equation (4)]

The coefficients of δq_k are known as the components of forces corresponding to generalised co-ordinates q_k and are denoted by Q'_k , i.e.,

$$Q'_k = \sum_i \mathbf{F}'_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} \quad \dots (27)$$

The Lagrange's equation in this case becomes

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} = Q'_k. \quad \dots (28)$$

In the above equation the conservative forces are included in the Lagrangian L ; while the non-conservative forces, i.e., the forces not derivable from a potential function V , are represented by Q'_k .

15.3. (b) Lagrange's equations for systems containing dissipative forces. Consider any system of particles containing dissipative or frictional forces. It is the experimental fact that in general the dissipative or frictional force is proportional to the velocity of the particles, i.e.,

$$\mathbf{F}_i^{(d)} = -\lambda_i \dot{\mathbf{r}}_i \quad \dots (1)$$

where $\mathbf{F}_i^{(d)}$ denotes the dissipative force, $\dot{\mathbf{r}}_i$ is the velocity of i th particle and λ_i is the corresponding constant of proportionality.

The dissipative or frictional forces, which are proportional to velocity, may be derived in terms of a new quantity, the *Rayleigh's dissipative function*, which is defined as

$$R = \frac{1}{2} \sum_i \lambda_i \dot{\mathbf{r}}_i^2, \quad \dots (2)$$

where the summation is taken over all the particles of the system, which gives

$$\frac{\partial R}{\partial \dot{\mathbf{r}}_i} = \lambda_i \dot{\mathbf{r}}_i = -\mathbf{F}_i^{(d)} \quad \text{from equation (1)}$$

or

$$\mathbf{F}_i^{(d)} = -\frac{\partial R}{\partial \dot{\mathbf{r}}_i} \quad \dots (3)$$

Lagrange's equation in generalised coordinates q_k is

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} = Q_k^{(d)} \quad \dots (4)$$

where $Q_k^{(d)}$ is component of generalised force.

In order to find the Lagrange's equation in generalised co-ordinates, we have to find the components of generalised force resulting from the dissipative or frictional force.

If $Q_k^{(d)}$ is the component of generalised force along q_k , then

$$Q_k^{(d)} = \sum_i \mathbf{F}_i^{(d)} \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} = - \sum_i \lambda_i \dot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} \quad [\text{from equation (1)}]$$

$$= - \sum_i \lambda_i \dot{\mathbf{r}}_i \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_k}$$

$$\left[\text{since } \frac{\partial \mathbf{r}_i}{\partial q_k} = \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_k} \text{ from equation (10) of section (11.3 a)} \right]$$

$$\text{or } Q_k^{(d)} = - \sum_i \lambda_i \frac{\partial}{\partial \dot{q}_k} \left(\frac{1}{2} \dot{\mathbf{r}}_i^2 \right) = - \frac{\partial}{\partial \dot{q}_k} \sum_i \left(-\frac{1}{2} \lambda_i \dot{\mathbf{r}}_i^2 \right)$$

$$\therefore Q_k^{(d)} = - \frac{\partial R}{\partial \dot{q}_k} \quad \dots (5)$$

[from equation (2)]

Therefore the Lagrange's equation for any system containing dissipative force is given by

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} + \frac{\partial R}{\partial \dot{q}_k} = 0. \quad \dots (6)$$

Hence the term $\frac{\partial R}{\partial \dot{q}_k}$ takes into account the dissipative forces.

15.3. (c) Applications of Lagrange's Equations

By the use of Lagrange's equations nearly all the dynamical problems can be treated by a proper choice of the Lagrangian function L . Therefore these equations are frequently chosen as

the fundamental equations of classical mechanics in place of Newton's laws. Due to this fact the Lagrange's equations in terms of properly chosen generalised coordinates represent the equations of motion of particle or system of particles. These Lagrange's equations are formulated as explained below :

1. The position vector (or cartesian coordinates) of particles are expressed as function of generalised coordinates according to transformation equations i.e.,

$$\begin{aligned} \mathbf{r}_i &= \mathbf{r}_i(q_1, q_2, \dots, q_k, \dots, q_f) \\ \text{or} \quad x_i &= x_i(q_1, q_2, \dots, q_k, \dots, q_f) \\ y_i &= y_i(q_1, q_2, \dots, q_k, \dots, q_f) \\ z_i &= z_i(q_1, q_2, \dots, q_k, \dots, q_f) \end{aligned}$$

2. The cartesian velocity components are expressed in terms of generalised velocities i.e.,

$$\dot{\mathbf{r}}_i = \frac{\partial \mathbf{r}_i}{\partial q_k} \dot{q}_k \quad \text{or} \quad \dot{x}_i = \frac{\partial x_i}{\partial q_k} \dot{q}_k, \quad \dot{y}_i = \frac{\partial y_i}{\partial q_k} \dot{q}_k, \quad \dot{z}_i = \frac{\partial z_i}{\partial q_k} \dot{q}_k$$

3. The potential energy V is assumed to be function of generalised coordinates and is expressed in terms of generalised co-ordinates choosing an arbitrary reference level (or zero potential energy level) i.e.,

$$V = V(q_1, q_2, q_3, \dots, q_k, \dots, q_f)$$

4. The velocities are substituted in the expression for kinetic energy =

$$T = \frac{1}{2} \sum_{i=1}^n m_i \dot{\mathbf{r}}_i^2$$

It is to be noted that in the generalised coordinates the kinetic energy is the function of both q_k and $\dot{q}_k$.

5. The equation $L = T - V$ is formed.

6. The partial derivatives $\frac{\partial L}{\partial q_k}$ and $\frac{\partial L}{\partial \dot{q}_k}$ are found.

7. Lagrange's equations are formed according to the number of generalised coordinates.

SOLVED EXAMPLES

Ex. 1. Show that the kinetic energy of a system of particles can always be written as the sum of three homogeneous functions of the generalised velocities.

$$T = T_0 + T_1 + T_2$$

i.e., where T_0 is independent of generalised velocities, T_1 is linear in velocities and T_2 is quadratic in velocities. (Rohilkhand 2000, 1992)

Solution : To write the Lagrangian of a system it is desirable to transform the kinetic energy and potential energy expressions from cartesian to generalised coordinates. Here the problem is to convert the kinetic energy from cartesian coordinates to generalised coordinates.

Consider a system of particles. let there be i th particle of mass m_i having position vector $\mathbf{r}_i$ at any instant t . Then kinetic energy of system

$$T = \frac{1}{2} \sum_i m_i \dot{\mathbf{r}}_i^2 \quad \dots (1)$$

If $(q_1, q_2, \dots, q_k, \dots, q_n, t)$ are generalised coordinates, then transformation equation is

$$\mathbf{r}_i = \mathbf{r}_i(q_1, q_2, \dots, q_k, \dots, q_n, t) \quad \dots (2)$$

We have

$$\dot{\mathbf{r}}_i = \sum_k \frac{\partial \mathbf{r}_i}{\partial q_k} \dot{q}_k + \frac{\partial \mathbf{r}_i}{\partial t} \quad \dots (3)$$

Substituting this value in (1), we get

$$\begin{aligned} T &= \frac{1}{2} \sum_i m_i \left[\sum_k \frac{\partial \mathbf{r}_i}{\partial q_k} \dot{q}_k + \frac{\partial \mathbf{r}_i}{\partial t} \right]^2 \\ &= \frac{1}{2} \sum_i m_i \left[\sum_k \sum_l \frac{\partial \mathbf{r}_i}{\partial q_k} \frac{\partial \mathbf{r}_i}{\partial q_l} \dot{q}_k \dot{q}_l + 2 \sum_k \frac{\partial \mathbf{r}_i}{\partial q_k} \dot{q}_k \cdot \frac{\partial \mathbf{r}_i}{\partial t} + \left(\frac{\partial \mathbf{r}_i}{\partial t} \right)^2 \right] \\ &= \sum_i \sum_k \sum_l \frac{1}{2} m_i \frac{\partial \mathbf{r}_i}{\partial q_k} \frac{\partial \mathbf{r}_i}{\partial q_l} \dot{q}_k \dot{q}_l + \sum_i \sum_k m_i \frac{\partial \mathbf{r}_i}{\partial q_k} \frac{\partial \mathbf{r}_i}{\partial t} \dot{q}_k + \sum_i \frac{1}{2} m_i \left(\frac{\partial \mathbf{r}_i}{\partial t} \right)^2 \\ &= \sum_{k,l} a_{kl} \dot{q}_k \dot{q}_l + \sum_k a_k \dot{q}_k + T_0 \end{aligned}$$

where $a_{kl} = \sum_i \frac{1}{2} m_i \frac{\partial \mathbf{r}_i}{\partial q_k} \frac{\partial \mathbf{r}_i}{\partial q_l}$, $a_k = \sum_i m_i \frac{\partial \mathbf{r}_i}{\partial q_k} \frac{\partial \mathbf{r}_i}{\partial t}$, $T_0 = \sum_i \frac{1}{2} m_i \left(\frac{\partial \mathbf{r}_i}{\partial t} \right)^2 \quad \dots (4)$

Now equation (4) may be written as $T = T_2 + T_1 + T_0$

where $T_2 = \sum_{k,l} a_{kl} \dot{q}_k \dot{q}_l$ is quadratic in generalised velocities

$T_1 = \sum_k a_k \dot{q}_k$ is linear in generalised velocities

and T_0 is independent of generalised velocities.

Ex. 2. Set up the Lagrangian for a simple pendulum and hence obtain equation describing its motion. Also find the time period of motion.

(Agra 2010, 09, Meerut 2003, Rohilkhand 2004, Garhwal 1998)

Solution. Simple pendulum

A Simple pendulum consists of a point mass 'm' suspended from a rigid support. When the bob is taken aside a little from equilibrium position and released it executes S.H.M. As the distance of point mass from suspension point S remains fixed, therefore only generalised coordinate is θ . If l is the length of simple pendulum, then

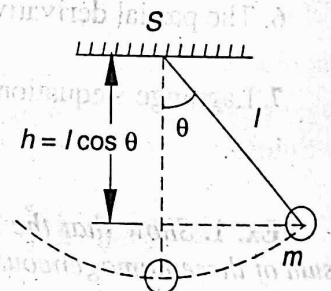


Fig. 15-2.

$$\text{Kinetic energy of mass, } T = \frac{1}{2} m \dot{x}^2 = \frac{1}{2} m (l \dot{\theta})^2 = \frac{1}{2} m l^2 \dot{\theta}^2$$

Taking reference level for zero potential energy as a horizontal plane passing through point of suspension S, the potential energy of mass,

$$V = -mgh = -mg l \cos \theta$$

$$\therefore \text{Lagrangian, } L = T - V = \frac{1}{2} m l^2 \dot{\theta}^2 + mgl \cos \theta$$

$$\therefore \frac{\partial L}{\partial \theta} = -mgl \sin \theta, \quad \frac{\partial L}{\partial \dot{\theta}} = m l^2 \dot{\theta}$$

Lagrange's equation is

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = 0$$

$$\begin{aligned}
 \text{or} \quad & \frac{d}{dt}(ml^2 \dot{\theta}) + mgl \sin \theta = 0 \\
 \text{or} \quad & ml^2 \ddot{\theta} + mgl \sin \theta = 0 \\
 \text{or} \quad & \ddot{\theta} = -\frac{g}{l} \sin \theta \quad \dots(1)
 \end{aligned}$$

This gives acceleration of point mass from Lagrange's equation.

For small angular displacement, $\sin \theta = \theta$

$$\therefore \ddot{\theta} = -\frac{g}{l} \theta \quad \dots(2)$$

Its solution is

$$\theta = \theta_0 \sin \left(\sqrt{\frac{g}{l}} t + \phi \right) \quad \dots(3)$$

where θ_0 is maximum angular amplitude and ϕ is initial phase

$$\text{Standard equation of SHM is } \ddot{\theta} = -\omega^2 \theta \quad \dots(4)$$

where ω is angular frequency

$\therefore$ Comparing (2) and (4)

$$\omega = \sqrt{\frac{g}{l}}$$

Time period

$$T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{l}{g}}$$

Ex. 3. For a mass m attached to a spring of force constant K , at any instant, the kinetic energy is given by $\frac{1}{2}mv^2$ and potential energy by $\frac{1}{2}Kx^2$, where v is a instantaneous velocity and x is displacement. Set up the Lagrangian of system and show that the dynamical equation of motion of mass is given by $a = -\omega^2 x$ where a is acceleration and $\omega^2 = \frac{K}{m}$ (Garhwal 1991)

Solution. The kinetic energy $T = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{x}^2$

$$\text{Potential energy } V = \frac{1}{2}Kx^2$$

$$\therefore \text{Lagrangian } L = T - V = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}Kx^2 \quad \dots(1)$$

The Lagrangian equation in coordinate x is

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = 0 \quad \dots(2)$$

$$\text{From (1), } \frac{\partial L}{\partial x} = -\frac{1}{2}K(2x) = -Kx$$

$$\text{and } \frac{\partial L}{\partial \dot{x}} = \frac{1}{2}m(2\dot{x}) = m\dot{x}$$

Substituting these values in (2), we get

$$\frac{d}{dt}(m\dot{x}) + Kx = 0$$

$$\text{or } m\ddot{x} + Kx = 0$$

$$\text{or } \ddot{x} + \frac{K}{m}x = 0$$

As $\ddot{x} = \frac{d^2x}{dt^2}$ = acceleration, so equation of motion is $a + \frac{K}{m}x = 0$

$$\text{or } a = -\omega^2x \text{ where } \omega^2 = \frac{K}{m}$$

This is the required result.

Ex. 4. (The Atwood's Machine). Obtain the equation of motion of a system of two particles of unequal masses connected by an inextensible string passing over a small smooth pulley. (Meerut 1992, Agra 1993)

Solution. The length of string, $PQ = l$,
mass of $P = m_1$ and mass of $Q = m_2$

Let $PA = x$, $QA = l - x$

Obviously, x is only independent variable.

Kinetic energy of mass $m_1 = \frac{1}{2} m_1 \dot{x}^2$

Kinetic energy of mass $m_2 = \frac{1}{2} m_2 \dot{x}^2$

$\therefore$ Total Kinetic energy of system $= \frac{1}{2} (m_1 + m_2) \dot{x}^2$

Considering the reference level as the horizontal place passing through A, the potential energies of masses m_1 and m_2 are respectively

$$-m_1gx \text{ and } -m_2g(l-x)$$

$\therefore$ Total potential energy of system,

$$V = -m_1gx - m_2g(l-x)$$

$\therefore$ The Lagrangian,

$$L = T - V = \frac{1}{2} (m_1 + m_2) \dot{x}^2 + m_1gx + m_2g(l-x) \quad \dots (1)$$

$$\text{So that } \left. \begin{aligned} \frac{\partial L}{\partial x} &= (m_1 - m_2)g \\ \frac{\partial L}{\partial \dot{x}} &= (m_1 + m_2)\dot{x} \end{aligned} \right\} \quad \dots (2)$$

$\therefore$ The Lagrange's equation in terms of generalised coordinate x is

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = 0$$

$$\text{or } \frac{d}{dt} \{ (m_1 + m_2) \dot{x} \} - (m_1 - m_2)g = 0$$

$$\text{which yields acceleration } \ddot{x} = \frac{m_1 - m_2}{m_1 + m_2} g \quad \dots (3)$$

Ex. 5. A bead sliding on a uniformly rotating wire. A bead is sliding on a uniformly rotating wire in a force free space. Find the resulting motion. (Agra 1980)

Solution. Let us suppose that a bead is sliding on a uniformly rotating wire in a force free space. In this case the constraint is time dependent. The bead may be treated as a particle. If

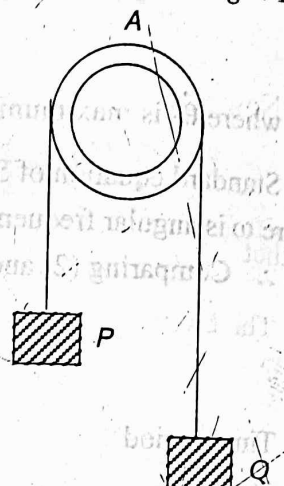


Fig. 15.3

(x, y) and (r, θ) are respectively the cartesian and polar co-ordinates of the particle at any instant t , then the transformation equations are given by

$$\left. \begin{aligned} x &= r \cos \theta = r \cos \omega t \\ y &= r \sin \theta = r \sin \omega t \end{aligned} \right\}, \text{ (since } \theta = \omega t), \quad \dots (1)$$

where ω is angular velocity of rotation.

The kinetic energy T of the bead (or particle) is given by

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2). \quad \dots (2)$$

since from equation (1),

$$\dot{x} = \dot{r} \cos \omega t - r \omega \sin \omega t,$$

$$\dot{y} = \dot{r} \sin \omega t + r \omega \cos \omega t.$$

Now equation (2) gives

$$T = \frac{1}{2} m (\dot{r}^2 + r^2 \omega^2) \quad \dots (3)$$

So that

$$\frac{\partial T}{\partial r} = m r \omega^2 \text{ and } \frac{\partial T}{\partial \dot{r}} = m \dot{r}. \quad \dots (4)$$

The Lagrange's equation in terms of r is given by

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{r}} \right) - \frac{\partial T}{\partial r} = Q_r$$

In the case under consideration $Q_r = 0$, since the bead is sliding in a *force-free* space. Therefore the Lagrange's equation becomes

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{r}} \right) - \frac{\partial T}{\partial r} = 0$$

or

$$\frac{d}{dt} (m \dot{r}) - m r \omega^2 = 0 \text{ from equation (4)}$$

or

$$m \ddot{r} - m r \omega^2 = 0$$

or

$$\ddot{r} = r \omega^2, \quad \dots (5)$$

which represents the centripetal acceleration due to which the bead moves outwards.

Ex. 6. Two particles of masses m_1 and m_2 move under the action of their gravitational interaction, find the Lagrangian equations of motion of the particles. (Madras 1995)

Solution. Assuming the line of interaction of particles, along x -axis, let coordinates of masses m_1 and m_2 be x_1 and x_2 respectively. Then separation between masses $x = x_2 - x_1$. Kinetic energy of system (both masses)

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2$$

$$\text{Potential energy of system, } V = -\frac{G m_1 m_2}{x_2 - x_1}$$

$$\therefore \text{Lagrangian of system, } L = T - V$$

$$= \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 + \frac{G m_1 m_2}{x_2 - x_1} \quad \dots (1)$$

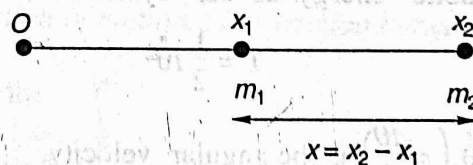


Fig. 15.4

$$\frac{\partial L}{\partial \dot{x}_1} = m_1 \dot{x}_1$$

$$\frac{\partial L}{\partial \dot{x}_2} = m_2 \dot{x}_2$$

$$\frac{\partial L}{\partial x_1} = -\frac{G m_1 m_2}{(x_2 - x_1)^2}$$

$$\frac{\partial L}{\partial x_2} = \frac{G m_1 m_2}{(x_2 - x_1)^2}$$

$\therefore$ Lagrangian equation for mass m_1 becomes

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}_1} \right) - \frac{\partial L}{\partial x_1} = 0$$

$$\Rightarrow \frac{d}{dt} (m_1 \dot{x}_1) + \frac{G m_1 m_2}{(x_2 - x_1)^2} = 0$$

or $m_1 \ddot{x}_1 = -\frac{G m_1 m_2}{(x_2 - x_1)^2}$ or $\ddot{x}_1 = -\frac{G m_2}{x^2}$... (2)

Lagrangian equation for mass m_2 is

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}_2} \right) - \frac{\partial L}{\partial x_2} = 0 \Rightarrow \frac{d}{dt} (m_2 \dot{x}_2) - \frac{G m_1 m_2}{(x_2 - x_1)^2} = 0$$

or $m_2 \ddot{x}_2 = \frac{G m_1 m_2}{(x_2 - x_1)^2} \Rightarrow \ddot{x}_2 = \frac{G m_1}{x^2}$... (3)

Equation (2) and (3) yield required Lagrange's equations

Ex. 7. Use Lagrange's equations to find the equation of motion of a compound pendulum which oscillates in a vertical plane about a fixed horizontal axis. Hence find the period of oscillation of the compound pendulum.

Solution. Let G be the centre of mass of the compound pendulum and O be its intersection with horizontal axis of rotation. Also let θ be instantaneous angle which OG makes with the vertical axis OA and I be the moment of inertia of the body about the axis of rotation.

Kinetic energy of the system is

$$T = \frac{1}{2} I \dot{\theta}^2$$

where $\dot{\theta} \left(= \frac{d\theta}{dt} \right)$ is the angular velocity.

Considering the horizontal plane passing through O as reference level, the potential energy of the system is given by

$$V = -mg(OA), \text{ where } m \text{ is the mass of the pendulum}$$

$$= -mg(OG) \cos \theta$$

$$= -mgl \cos \theta.$$

(where $OG = l$).

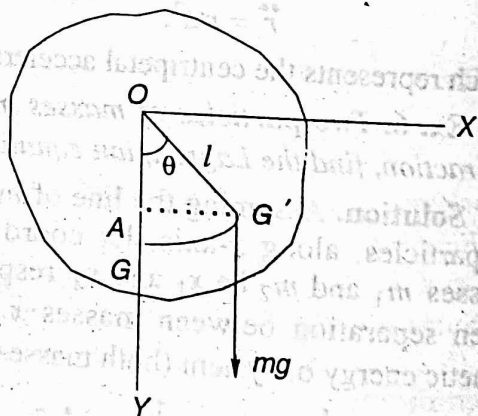


Fig. 15-5

The Lagrangian L is written as

$$L = T - V$$

$$= \frac{1}{2} I \dot{\theta}^2 + mgl \cos \theta, \quad \dots(1)$$

which gives $\frac{\partial L}{\partial \theta} = -mgl \sin \theta$ and $\frac{\partial L}{\partial \dot{\theta}} = I \dot{\theta}$... (2)

The Lagrangian equation or equation of motion in the generalised co-ordinate θ is

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = 0$$

or $\frac{d}{dt} (I \dot{\theta}) + mgl \sin \theta = 0$ from (2)

or $I \ddot{\theta} + mgl \sin \theta = 0$... (3)

or $\ddot{\theta} + \frac{mgl}{I} \sin \theta = 0$ (4)

If the amplitude of oscillation is small, then $\sin \theta \approx \theta$, so the equation of motion (4) takes the form

$$\ddot{\theta} + \frac{mgl}{I} \theta = 0. \quad \dots(5)$$

Therefore the angular frequency of the motion is

$$\omega = \sqrt{\left(\frac{mgl}{I} \right)} \quad \dots(6)$$

Hence period of oscillation of the compound pendulum is

$$T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{I}{mgl}} \quad \dots(7)$$

If k is radius of gyration of a body about an axis through centre of gravity G then

$$I = mk^2 + ml^2$$

$$T = 2\pi \sqrt{\frac{m(k^2 + l^2)}{mgl}} = 2\pi \sqrt{\frac{\left(\frac{k^2}{l} + l \right)}{g}}$$

Ex. 8. A bead slides on a wire in the shape of a cycloid described by equations

$$x = a(\theta - \sin \theta),$$

$$y = a(1 - \cos \theta),$$

where $0 \leq \theta \leq 2\pi$.

Find (a) the Lagrangian function, and (b) the equation of motion. [The friction between the bead and the wire may be assumed negligible].

Solution. (a) In figure 15-6, P is the position of the bead (assumed as a particle) of mass m at any instant t . Let the co-ordinates of P be (x, y) . The path of the bead (cycloid) is represented by the equations

$$\begin{cases} x = a(\theta - \sin \theta) \\ y = a(1 - \cos \theta) \end{cases} \quad \dots(1)$$

The kinetic energy T of the bead is given by

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2)$$

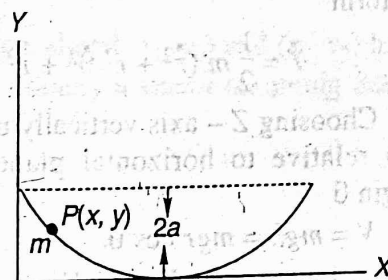


Fig. 15-6

$$\begin{aligned}
 &= \frac{1}{2} m \{a^2 (\dot{\theta} - \cos \theta \dot{\theta})^2 + a^2 (-\sin \theta \dot{\theta})^2\} \\
 &= \frac{1}{2} m a^2 \dot{\theta}^2 \{2 - 2 \cos \theta\} \\
 &= m a^2 \dot{\theta}^2 (1 - \cos \theta), \quad \dots (2)
 \end{aligned}$$

Taking x-axis as reference level, the potential energy of the bead is given by

$$V = mgy = mga (1 + \cos \theta) \quad \dots (3)$$

from equation (1).

The Lagrangian function L be written as

$$\begin{aligned}
 L &= T - V \\
 &= m a^2 \dot{\theta}^2 (1 - \cos \theta) - m g a (1 + \cos \theta) \quad \dots (4)
 \end{aligned}$$

[from equations (2) and (3)].

(b) The Lagrange's equation or equation of motion is

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = 0. \quad \dots (5)$$

Equation (3) gives

$$\frac{\partial L}{\partial \dot{\theta}} = 2 m a^2 \dot{\theta} (1 - \cos \theta)$$

and

$$\frac{\partial L}{\partial \theta} = 2 m a^2 \dot{\theta}^2 \sin \theta + m g a \sin \theta.$$

With these substitutions equation (5) becomes

$$\frac{d}{dt} \{2 m a^2 \dot{\theta} (1 - \cos \theta)\} - (2 m a^2 \dot{\theta}^2 \sin \theta + m g a \sin \theta) = 0$$

$$\text{or } 2 m a^2 \dot{\theta} (1 - \cos \theta) + 2 m a^2 \sin \theta \dot{\theta} - m a^2 \dot{\theta}^2 \sin \theta - m g a \sin \theta = 0$$

$$\text{or } \dot{\theta} (1 - \cos \theta) + \frac{1}{2} \sin \theta \dot{\theta} - \frac{g}{2a} \sin \theta = 0.$$

This is the required equation of motion.

Ex. 9. Set up the equation for a spherical pendulum and derive the equation of motion.

(Meerut 1998)

Solution. Spherical Pendulum A spherical pendulum consists of bob constrained to move on the surface of a sphere. The position of the bob is located by spherical co-ordinates (r, θ, ϕ) . The distance r of bob from centre of sphere is always equal to radius of sphere; hence r is constant, so only generalised coordinates of the problem are θ and ϕ .

$$\text{KE of bob } T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

This expression in spherical coordinates takes the form

$$T = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2) \quad \dots (1)$$

Choosing Z -axis vertically upward; the P.E of bob relative to horizontal plane passing through origin O

$$V = mgh = mgr \cos \theta$$

$$\therefore \text{Lagrangian, } L = T - V$$

$$= \frac{1}{2} m r^2 (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) - mgr \cos \theta$$

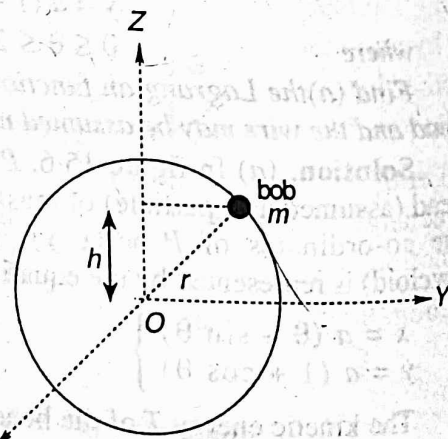


Fig. 15-7

This equation gives

$$\frac{\partial L}{\partial \dot{\theta}} = mr^2 \dot{\theta}, \quad \frac{\partial L}{\partial \theta} = mr^2 \sin \theta \cos \theta \dot{\phi}^2 + mgr \sin \theta$$

$$\frac{\partial L}{\partial \dot{\phi}} = mr^2 \sin^2 \theta \dot{\phi}, \quad \frac{\partial L}{\partial \phi} = 0$$

$\therefore$ Lagrange's equations of motion in coordinate θ and ϕ are

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = 0$$

$$\Rightarrow \frac{d}{dt} (mr^2 \dot{\theta}) - mr^2 \sin \theta \cos \theta \dot{\phi}^2 - mgr \sin \theta = 0 \quad \dots (2)$$

$$\text{and } \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\phi}} \right) - \frac{\partial L}{\partial \phi} = 0$$

$$\Rightarrow \frac{d}{dt} (mr^2 \sin^2 \theta \dot{\phi}) = 0 \quad \dots (3)$$

Equations (2) and (3) may be simplified as

$$mr^2 \ddot{\theta} - mr^2 \sin \theta \cos \theta \dot{\phi}^2 - mgr \sin \theta = 0 \quad \dots (4)$$

$$\text{and } \frac{d}{dt} (\sin^2 \theta \dot{\phi}) = 0 \quad \dots (5)$$

These are required equations of motion for a spherical pendulum.

Ex. 10. Obtain the Lagrangian and equations of motion for a double pendulum vibrating in a vertical plane. (Rohilkhand 1999, 94, 89, 87, 83, 81)

Solution. A double pendulum consists of two masses connected by an inextensible light rod. The system of two masses and the rod is suspended by another inextensible and weightless rod fastened to one of the masses as shown in figure (15.8). In another way a double pendulum may be described as follows:

As the name indicates a double pendulum may be supposed to be formed of two pendula oscillating in the same plane; one has a bob of mass m_1 suspended from a flat hinge at O by an inextensible and weightless rod of length l_1 , while the other has a rod of mass m_2 suspended from another hinge in mass m_1 by a similar rod of length l_2 .

Let θ_1 and θ_2 be the deflections of the pendula from the vertical and (x_1, y_1) and (x_2, y_2) be the rectangular co-ordinates of masses m_1 and m_2 respectively are any instant t assuming that the motion is confined in the XY - plane.

From figure 15.8,

$$\left. \begin{aligned} x_1 &= l_1 \sin \theta_1 \\ y_1 &= l_1 \cos \theta_1 \\ x_2 &= l_1 \sin \theta_1 + l_2 \sin \theta_2 \\ y_2 &= l_1 \cos \theta_1 + l_2 \cos \theta_2 \end{aligned} \right\} \quad \dots (1)$$

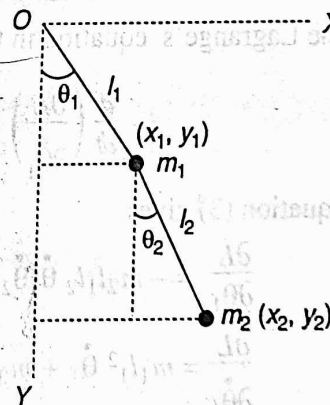


Fig 15.8

from which the velocity components are given by

$$\left. \begin{aligned} \dot{x}_1 &= l_1 \cos \theta_1 \dot{\theta}_1, \\ \dot{y}_1 &= -l_1 \sin \theta_1 \dot{\theta}_1, \\ \dot{x}_2 &= l_1 \cos \theta_1 \dot{\theta}_1 + l_2 \cos \theta_2 \dot{\theta}_2, \\ \dot{y}_2 &= -l_1 \sin \theta_1 \dot{\theta}_1 - l_2 \sin \theta_2 \dot{\theta}_2 \end{aligned} \right\} \dots (2)$$

Kinetic energy of the system is

$$\begin{aligned} T &= \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2) \\ &= \frac{1}{2} m_1 \{l_1^2 \dot{\theta}_1^2 + (-l_1 \sin \theta_1 \dot{\theta}_1)^2\} \\ &\quad + \frac{1}{2} m_2 \{(l_1 \cos \theta_1 \dot{\theta}_1 + l_2 \cos \theta_2 \dot{\theta}_2)^2 + (-l_1 \sin \theta_1 \dot{\theta}_1 - l_2 \sin \theta_2 \dot{\theta}_2)^2\} \\ &= \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 \{l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos (\theta_1 - \theta_2)\}. \end{aligned}$$

Taking the reference level as a horizontal plane passing through origin O , i.e., the point of suspension O , the potential energy of the system is given by

$$\begin{aligned} V &= -m_1 g y_1 - m_2 g y_2 \\ &= -m_1 g l_1 \cos \theta_1 - m_2 g (l_1 \cos \theta_1 + l_2 \cos \theta_2) \end{aligned}$$

The Lagrangian L is given by

$$\begin{aligned} L = T - V &= \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 (l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos (\theta_1 - \theta_2)) \\ &\quad + m_1 g l_1 \cos \theta_1 + m_2 g (l_1 \cos \theta_1 + l_2 \cos \theta_2) \end{aligned} \dots (3)$$

The Lagrange's equation in the variable θ_1 is

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}_1} \right) - \frac{\partial L}{\partial \theta_1} = 0. \dots (4)$$

Equation (3) gives

$$\begin{aligned} \frac{\partial L}{\partial \dot{\theta}_1} &= m_1 l_1 \dot{\theta}_1 + m_2 l_1 \dot{\theta}_1 + m_2 l_1 l_2 \dot{\theta}_2 \cos (\theta_1 - \theta_2) \\ \frac{\partial L}{\partial \theta_1} &= -m_1 g l_1 \sin \theta_1 - m_2 g l_1 \sin \theta_1 - m_2 l_1 l_2 \dot{\theta}_2 \sin (\theta_1 - \theta_2) \end{aligned}$$

With these substitutions, equation (4) becomes

$$\begin{aligned} \frac{d}{dt} \{m_1 l_1 \dot{\theta}_1 + m_2 l_1 \dot{\theta}_1 + m_2 l_1 l_2 \dot{\theta}_2 \cos (\theta_1 - \theta_2)\} &+ m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin (\theta_1 - \theta_2) \\ &+ m_1 g l_1 \sin \theta_1 + m_2 g l_1 \sin \theta_1 = 0 \\ \text{or } (m_1 + m_2) l_1^2 \ddot{\theta}_1 &+ m_2 l_1 l_2 \ddot{\theta}_2 \cos (\theta_1 - \theta_2) - m_2 l_1 l_2 \dot{\theta}_2 \sin (\theta_1 - \theta_2) (\dot{\theta}_1 - \dot{\theta}_2) \\ &+ m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin (\theta_1 - \theta_2) + m_1 g l_1 \sin \theta_1 + m_2 g l_1 \sin \theta_1 = 0 \\ \text{or } (m_1 + m_2) l_1^2 \ddot{\theta}_1 &+ m_2 l_1 l_2 \ddot{\theta}_2 \cos (\theta_1 - \theta_2) + m_2 l_1 l_2 \dot{\theta}_2^2 \sin (\theta_1 - \theta_2) \\ &+ (m_1 + m_2) g l_1 \sin \theta_1 = 0 \end{aligned} \dots (5)$$

The Lagrange's equation in variable θ_2 is

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}_2} \right) - \frac{\partial L}{\partial \theta_2} = 0. \quad \dots (6)$$

From equation (3) we have

$$\frac{\partial L}{\partial \dot{\theta}_2} = m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \dot{\theta}_1 \cos(\theta_1 - \theta_2),$$

$$\frac{\partial L}{\partial \theta_2} = m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - m_2 g l_2 \sin \theta_2.$$

With these substitutions equation (6) becomes

$$\frac{d}{dt} \{ m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \dot{\theta}_1 \cos(\theta_1 - \theta_2) \} - m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) + m_2 g l_2 \sin \theta_2 = 0$$

$$\text{or } m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - m_2 l_1 l_2 \dot{\theta}_1 \sin(\theta_1 - \theta_2) (\dot{\theta}_1 - \dot{\theta}_2) - m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) + m_2 g l_2 \sin \theta_2 = 0$$

$$\text{or } m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - m_2 l_1 l_2 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) + m_2 g l_2 \sin \theta_2 = 0. \dots (7)$$

Equations (5) and (7) are required equations of motion.

Ex. 11. A particle is constrained to move in a plane under the influence of an attraction towards the origin proportional to the distance from it and also of a force perpendicular to the radius vector inversely proportional to the distance of the particle from the origin in anticlockwise direction. Find (a) the Lagrangian, and (b) the equations of motion.

(Rohilkhand 2003, Agra 1997)

Solution. This is an example of partly conservative and partly non-conservative system.

Let (r, θ) be the polar co-ordinates of the particle of mass m at any instant t . The kinetic energy of the particle is

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2), \quad \dots (1)$$

The force acting on the particle is

$$F = -kr \quad \dots (2)$$

where k is constant of proportionality.

But we know $F = -\frac{dV}{dr},$

where V is the potential energy

or $dV = -F dr.$

Therefore potential energy of the particle is

$$V = - \int F dr + A, \text{ where } A \text{ is constant of integration}$$

$$= \int kr dr + A$$

from equation (2)

$$= \frac{kr^2}{2} + A.$$

At the origin potential energy is zero, i.e., $V = 0$ at $r = 0$, thereby giving $A = 0$.

Then the potential energy is given by

$$V = \frac{kr^2}{2} \quad \dots (3)$$

The Lagrangian is

$$L = T - V$$

$$= \frac{1}{2} m(\dot{r}^2 + r^2 \dot{\theta}^2) - \frac{kr^2}{2} \quad \dots (4)$$

from which we have

$$\left[\begin{aligned} \frac{\partial L}{\partial \dot{r}} &= m \dot{r}, \quad \frac{\partial L}{\partial r} = mr\dot{\theta}^2 - kr \\ \frac{\partial L}{\partial \dot{\theta}} &= mr^2 \dot{\theta}, \quad \frac{\partial L}{\partial \theta} = 0 \end{aligned} \right] \quad \dots (5)$$

The Lagrange's equations of motion are

$$\left[\begin{aligned} \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{r}} \right) - \frac{\partial L}{\partial r} &= Q_r', \\ \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} &= Q_\theta', \end{aligned} \right] \quad \dots (6)$$

where Q_r' and Q_θ' are non-conservative generalised forces.

In this case $Q_r' = 0$, since non-conservative force is perpendicular to the radius vector $\mathbf{r}$,

$$Q_\theta' = F' \cdot \frac{\partial \mathbf{r}}{\partial \theta} \quad \text{[from equation (27) of article 15.3a]}$$

$$= rF' \quad \text{[since } \delta \mathbf{r} = r \delta \theta \hat{n} \text{ in a direction normal to radius vector]}$$

$$= \frac{r\lambda}{r} \quad \text{(since non-conservative force is inversely proportional to } r, \text{ i.e.,}$$

$$F' = \frac{\lambda}{r} \text{ where } \lambda \text{ is a constant)}$$

$$= \lambda$$

With these substitutions equations (6) become

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{r}} \right) - \frac{\partial L}{\partial r} = 0$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = \lambda.$$

Making substitution from (5) these equations of motion may be written as

$$m\ddot{r} - mr\dot{\theta}^2 + kr = 0$$

and

$$\frac{d}{dt} (mr^2 \dot{\theta}) = \lambda$$

Ex. 12. A particle of mass m is projected with initial velocity u at an angle α with the horizontal. Use Lagrange's equations to describe the motion of the projectile. The resistance of air may be neglected. (Rohilkhand 2003, 87)

Or

Obtain Lagrange's equations of motion for a projectile and solve them to show that the trajectory of a projectile is a parabola (neglect air resistance) (Rohilkhand 1989)

Solution. Let us suppose that the motion takes place in the XY plane. Also suppose that the horizontal line is represented by axis of X .

If $P(x, y)$ represents the position of the particle at any instant t , its kinetic energy is

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) \quad \dots(1)$$

The potential energy of the particle, taking X -axis as the reference level is given by

$$V = mgy \quad \dots(2)$$

Thus the Lagrangian L is

$$L = T - V = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) - mgy \quad \dots(3)$$

[from (1) and (2)]

The Lagrange's equations are

$$\left. \begin{aligned} \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} &= 0 \\ \text{and} \quad \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{y}} \right) - \frac{\partial L}{\partial y} &= 0 \end{aligned} \right\} \quad \dots(4)$$

From equation (3), we have

$$\begin{aligned} \frac{\partial L}{\partial \dot{x}} &= m\dot{x}, \quad \frac{\partial L}{\partial x} = 0 \\ \frac{\partial L}{\partial \dot{y}} &= m\dot{y}, \quad \frac{\partial L}{\partial y} = -mg. \end{aligned}$$

With these substitutions the Lagrange's equations (or equations of motion) are given by

$$\frac{d}{dt} (m\dot{x}) = m\ddot{x} = 0$$

$$\text{and} \quad \frac{d}{dt} (m\dot{y}) + mg = m\ddot{y} + mg = 0.$$

$$\left. \begin{aligned} \ddot{x} &= 0 \quad \dots (a) \\ \ddot{y} &= -g \quad \dots (b) \end{aligned} \right\} \quad \dots(5)$$

Integrating equation (5a), we get

$$\frac{dx}{dt} = A (= \text{constant}) \quad \dots(6)$$

or

$$dx = A dt.$$

The integration of above equation yields

$$x = At + B, B \text{ being the constant of integration.} \quad \dots(7)$$

The initial conditions are :

$$\left. \begin{aligned} x &= 0 & \text{at } t &= 0, \\ \dot{x} &= u \cos \alpha & \text{at } t &= 0, \end{aligned} \right\} \quad \dots(8)$$

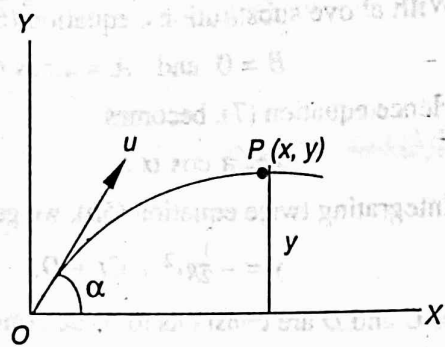


Fig. 15-9

With above substitutions, equation (6) gives

$$B = 0 \text{ and } A = u \cos \alpha. \quad \dots(9)$$

Hence equation (7), becomes

$$x = u \cos \alpha t. \quad \dots(10)$$

Integrating twice equation (5b), we get

$$y = -\frac{1}{2}gt^2 + Ct + D, \quad \dots(11)$$

where C and D are constants to be determined by initial conditions which are

$$\left. \begin{aligned} y &= 0 & \text{at } t &= 0, \\ \dot{y} &= u \sin \alpha & \text{at } t &= 0. \end{aligned} \right\} \quad \dots(12)$$

With these conditions equation (10) gives

$$D = 0 \text{ and } C = u \sin \alpha.$$

Hence equation (11) may be written as

$$y = u \sin \alpha \cdot t - \frac{1}{2}gt^2. \quad \dots(12)$$

Substituting the value of t from equation (10) in the above equation, we get we get

$$y = x \tan \alpha - \frac{1}{2}g \frac{x^2}{u^2 \cos^2 \alpha} \quad \dots(13)$$

This represents that *the path of the projectile is a parabola.*

Ex. 13. Obtain the equation of motion for a particle falling vertically under the influence of gravity when frictional forces obtainable from a dissipative function $\frac{1}{2}kv^2$ are present. Integrate the equation to obtain the velocity as a function of time. Hence show that the maximum possible velocity for fall from rest is $v = \frac{mg}{k}$.

Solution. The Lagrange's equations for system containing the frictional forces obtainable from a dissipative function R are given by

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} + \frac{\partial R}{\partial \dot{q}_k} = 0.$$

Let y be the height of the particle from the earth at any instant t . The kinetic energy is

$$T = \frac{1}{2}m\dot{y}^2 \quad \dots(2)$$

The potential energy is $V = mgy$,

Here

$$R = \frac{1}{2}kv^2. \quad \dots(3)$$

Therefore

$$\frac{\partial R}{\partial \dot{y}} = \frac{\partial R}{\partial v} = kv = ky. \quad \dots(4)$$

The Lagrangian is

$$L = T - V = \frac{1}{2}m\dot{y}^2 - mgy.$$

from which we have

$$\frac{\partial L}{\partial \dot{y}} = m\dot{y}, \text{ and } \frac{\partial L}{\partial y} = -m.g \quad \dots(5)$$

The Lagrange's equation or equation of motion is given by

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{y}} \right) - \frac{\partial L}{\partial y} + \frac{\partial R}{\partial \dot{y}} = 0$$

or

$$m\ddot{y} + mg + k\dot{y} = 0$$

[From (4) and (5)] ... (6)

The above equation may be written as

$$m\ddot{y} + k\dot{y} = -mg.$$

Integrating with respect to time t , we have

$$\int m\ddot{y} dt + \int k\dot{y} dt = - \int mg dt + C, \quad C \text{ being constant of integration}$$

or

$$m\dot{y} + ky = -mgt + C$$

or

$$mv + ky = -mgt + C. \quad \dots (7)$$

According to initial condition, at $t = 0$, $v = 0$, $y = h$ (say).

With this substitution equation (7) becomes

$$mv = -mgt + k(h - y)$$

or

$$v = -gt + \frac{k}{m}(h - y).$$

Differentiating above equation with respect to time, we get

$$\frac{dv}{dt} = -g + \frac{k}{m} \left(-\frac{dy}{dt} \right).$$

The velocity will be maximum when $\frac{dv}{dt} = 0$; therefore maximum velocity $\left(\frac{dy}{dt} = v \right)$ is given by

$$-g - \frac{k}{m}v = 0$$

or

$$v = \frac{mg}{k}$$

Ex. 14. A particle is moving under the influence of a central force $F = -\frac{kr}{r^3}$. Set up the Lagrangian and hence derive the equation of motion. (Rohilkhand 1998, Garhwal 1990)

Or

A particle of mass m is moving in a plane under an inverse square law attractive force. Set up the Lagrangian and hence obtain the equation describing its motion. (Meerut 1997)

Solution. Let (r, θ) be the plane polar co-ordinates of the particle of mass m . Then the kinetic energy of the particle T is given by

$$T = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2)$$

If F is the force experienced by the particle, then

$$\text{As } F \propto \frac{1}{r^2} = -\frac{k}{r^2} \hat{r}$$

as unit vector $\hat{r} = \frac{\mathbf{r}}{r} \therefore \mathbf{F} = -\frac{kr}{r^3}$ or $F = -\frac{k}{r^2}$ (numerically)

where k is the constant of proportionality.

We know $F = -\frac{dV}{dr}$

or $dV = -F dr$.

Potential energy V is given by

$$V = - \int_{\infty}^r F dr = \int_{\infty}^r \frac{k}{r^2} dr = \left[-\frac{k}{r} \right]_{\infty}^r = -\frac{k}{r}$$

The Lagrangian L is given by

$$L = T - V = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) + \frac{k}{r},$$

which gives $\left. \begin{aligned} \frac{\partial L}{\partial r} &= m r \dot{\theta}^2, \quad \frac{\partial L}{\partial r} = m r \dot{\theta}^2 - \frac{k}{r^2}, \\ \text{and} \quad \frac{\partial L}{\partial \dot{\theta}} &= m r^2 \dot{\theta}; \quad \frac{\partial L}{\partial \dot{\theta}} = 0; \end{aligned} \right\}$

... (3)

The Lagrangian equation (or equation of motion) in the variable r is given

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{r}} \right) - \frac{\partial L}{\partial r} = 0$$

or

$$\frac{d}{dt} (m \dot{r}) - \left(m r \dot{\theta}^2 - \frac{k}{r^2} \right) = 0$$

or

$$m \ddot{r} - m r \dot{\theta}^2 + \frac{k}{r^2} = 0. \quad \dots (3)$$

The Lagrangian equation in the variable θ is given by

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = 0$$

or

$$\frac{d}{dt} (m r^2 \dot{\theta}) = 0$$

— from equation (2)

or

$$m r^2 \ddot{\theta} + 2 m r \dot{r} \dot{\theta} = 0. \quad \dots (4)$$

Ex. 15. A particle of mass M moves on a plane in the field force given by (in polar co-ordinates).

$$F = -I_r k r \cos \theta,$$

where k is constant and I_r is the radial unit vector.

(a) Will the angular momentum of the particle about the origin be conserved? Justify your statement.

(b) Obtain the differential equation of the orbit of the particle. (Agra 1998)

Solution. If (x, y) and (r, θ) are the cartesian and polar co-ordinates of the particle of mass M in the given plane, the kinetic energy of the particle is given by

$$\begin{aligned} T &= \frac{1}{2} M (\dot{x}^2 + \dot{y}^2) \\ &= \frac{1}{2} M (\dot{r}^2 + r^2 \dot{\theta}^2). \end{aligned} \quad \dots (1)$$

since $x = r \cos \theta$, $y = r \sin \theta$.

(a) Equation (1) gives

$$\frac{\partial L}{\partial \dot{\theta}} = M r^2 \dot{\theta}; \quad \frac{\partial T}{\partial \dot{\theta}} = 0 \quad \dots (2)$$

The Lagrangian equation in terms of θ is

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\theta}} \right) - \frac{\partial T}{\partial \theta} = Q_{\theta} \quad \dots (3)$$

But $Q_{\theta} = 0$, since there is no force in the direction of θ .

Therefore the Lagrangian equation is

$$\frac{d}{dt} (Mr^2 \dot{\theta}) = 0, \quad \text{from (2) and (3)}$$

i.e.

$$Mr^2 \ddot{\theta} = \text{constant.}$$

Thus the angular momentum ($Mr^2 \dot{\theta}$) about the origin is conserved.

(b) From equation (1), we have

$$\frac{\partial T}{\partial \dot{r}} = Mr \dot{\theta}, \quad \frac{\partial T}{\partial r} = Mr \dot{\theta}^2 \quad \dots (4)$$

The Lagrangian equation in terms of r is

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{r}} \right) - \frac{\partial T}{\partial r} = Q_r \quad \dots (5)$$

But $Q_r = -kr \cos \theta$ (according to the given problem)

Putting the value of $\frac{\partial T}{\partial \dot{r}}$, $\frac{\partial T}{\partial r}$ from (4) and Q_r in equation (5), we

have $Mr \ddot{r} - Mr \dot{\theta}^2 = -kr \cos \theta \quad \dots (6)$

which gives the equation of the orbit of the particle.

15.3. (d) Lagrangian Formulation of Conservation Theorems

For the purpose let us first define *generalised momentum* and *cyclic co-ordinates*.

Generalised momentum

Consider a conservative system, i.e., a system in which the forces are derivable from a potential function V dependent on position only; then

$$\begin{aligned} \frac{\partial L}{\partial \dot{q}_k} &= \frac{\partial (T - V)}{\partial \dot{q}_k} = \frac{\partial T}{\partial \dot{q}_k} - \frac{\partial V}{\partial \dot{q}_k} \\ &= \frac{\partial T}{\partial \dot{q}_k} \quad [\text{because potential function } V \text{ depends only on position or } \partial V / \partial \dot{q}_k = 0] \\ &= \frac{\partial}{\partial \dot{q}_k} \sum_i \frac{1}{2} m_i \dot{q}_i^2 \\ &= m_k \dot{q}_k = p_k. \end{aligned}$$

p_k is called the *generalised momentum* associated with generalised co-ordinate q_k .

This the generalised momentum corresponding to the generalised co-ordinate q_k is defined as the quantity $\frac{\partial L}{\partial \dot{q}_k}$ and is represented by p_k

i.e.,

$$p_k = \frac{\partial L}{\partial \dot{q}_k}$$

Cyclic or ignorable co-ordinates,

If the Lagrangian of a system is not the function of a given co-ordinate q_k , then the co-ordinate q_k is said to be cyclic or ignorable.

In such case
$$\frac{\partial L}{\partial q_k} = 0. \quad \dots(2)$$

(i) **Conservation Theorem for Generalised Momentum.** According to Lagrange's equation,

$$-\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} = 0.$$

If the generalised co-ordinate q_k is cyclic or ignorable, then we have

$$\frac{\partial L}{\partial q_k} = 0 \text{ [from equation (2)]}$$

With this substitution equation (3) becomes

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) = 0$$

or
$$\frac{\partial L}{\partial \dot{q}_k} = \text{constant}$$

i.e.,
$$p_k = \text{constant [from equation (1)]}.$$

This gives the conservation theorem for generalised momentum which states that *the generalised momentum corresponding to a cyclic (or ignorable) co-ordinate is conserved.*

(ii) Conservation Theorem for Energy

Consider a conservative system, i.e., a system in which the forces are derivable from a potential function V dependent on position only. The Lagrangian L is given by

$$L = L(q_1, q_2, \dots, q_k, \dots, \dot{q}_1, \dot{q}_2, \dots, \dot{q}_k, \dots, t).$$

Let us assume that L does not depend upon time explicitly. Then

$$L = L(q_1, q_2, \dots, q_k, \dots, \dot{q}_1, \dot{q}_2, \dots, \dot{q}_k, \dots).$$

The total time derivative of L is given by

$$\begin{aligned} \frac{dL}{dt} &= \frac{\partial L}{\partial q_1} \dot{q}_1 + \frac{\partial L}{\partial q_2} \dot{q}_2 + \dots + \frac{\partial L}{\partial q_k} \dot{q}_k + \dots + \frac{\partial L}{\partial \dot{q}_1} \ddot{q}_1 + \frac{\partial L}{\partial \dot{q}_2} \ddot{q}_2 + \dots + \frac{\partial L}{\partial \dot{q}_k} \ddot{q}_k + \dots \\ &= \sum_k \frac{\partial L}{\partial q_k} \dot{q}_k + \sum_k \frac{\partial L}{\partial \dot{q}_k} \ddot{q}_k. \end{aligned} \quad \dots(4)$$

Lagrange's equation is

$$-\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} = 0$$

or
$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) = -\frac{\partial L}{\partial q_k}$$

With this substitution equation (4) becomes

$$\frac{dL}{dt} = \sum_k \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) \dot{q}_k + \sum_k \frac{\partial L}{\partial \dot{q}_k} \ddot{q}_k$$

or

$$\frac{dL}{dt} = \sum_k \frac{d}{dt} \left(\dot{q}_k \frac{\partial L}{\partial \dot{q}_k} \right)$$

or

$$\frac{d}{dt} \left(\sum_k \dot{q}_k \frac{\partial L}{\partial \dot{q}_k} - L \right) = 0 \quad \dots (5)$$

Thereby indicating that $\sum_k \dot{q}_k \frac{\partial L}{\partial \dot{q}_k} - L = \text{constant}$.

But

$$\frac{\partial L}{\partial \dot{q}_k} = \frac{\partial}{\partial \dot{q}_k} (T - V) = \frac{\partial T}{\partial \dot{q}_k} = \frac{\partial}{\partial \dot{q}_k} \left(\sum_i \frac{1}{2} m_i \dot{q}_i^2 \right)$$

because $T = \sum_i \frac{1}{2} m_i \dot{q}_i^2$, where m_i is the mass of i th particle.

$$\therefore \frac{\partial L}{\partial \dot{q}_k} = m_k \dot{q}_k$$

Substituting the value of $\frac{\partial L}{\partial \dot{q}_k}$ from above equation in (5), we get

$$\sum m_k \dot{q}_k^2 - L = \text{constant}$$

or

$$2T - L = \text{constant}$$

or

$$2T - (T - V) = \text{constant}$$

Therefore

$$T + V = \text{total energy} = \text{constant}$$

Thus the energy conservation theorem states that **if the Lagrangian function does not contain the time explicitly, the total energy of the conservative system is conserved.**

(iii) **Conservation Theorem for Linear Momentum.** Consider conservative system so that the potential energy V is dependent on position only and the kinetic energy is independent of position, i.e.

$$\frac{\partial V}{\partial \dot{q}_k} = 0 \text{ and } \frac{\partial T}{\partial q_k} = 0 \quad \dots (6)$$

The Lagrange's equation for such a system may be written as

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} = 0$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial (T - V)}{\partial q_k} = 0$$

i.e.

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) + \frac{\partial V}{\partial q_k} = 0 \quad [\text{using eqn. (6)}]$$

So

$$\dot{p}_k = \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) = - \frac{\partial V}{\partial q_k} = Q_k \quad \dots (7)$$

Now, if we show that the generalised force, Q_k represents the component of total force along the direction of translation of the system and generalised momentum p_k is the component of total linear momentum along the same direction, then eqn. (7) will represent the equation of motion for total linear momentum.

The generalised force Q_k is given by

$$Q_k = \sum_i \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k}$$

If $\hat{n}$ is the unit vector along the direction of translation of the system, then $\delta \mathbf{r}_i = \hat{n} \delta q_k$

i.e.
$$\frac{\partial \mathbf{r}_i}{\partial q_k} = \hat{n}$$

So

$$Q_k = \sum_i \mathbf{F}_i \cdot \hat{n} = \hat{n} \cdot \mathbf{F} \quad \dots (8)$$

which represents the component of total force along the direction of $\hat{n}$ (i.e. translation).

The kinetic energy, $T = \frac{1}{2} \sum_i m_i \dot{\mathbf{r}}_i^2$.

So the generalised momentum,

$$p_k = \frac{\partial L}{\partial \dot{q}_k} = \frac{\partial T}{\partial \dot{q}_k} = \frac{\partial}{\partial \dot{q}_k} \sum_i \frac{1}{2} m_i \dot{\mathbf{r}}_i^2 = \sum_i m_i \dot{\mathbf{r}}_i \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_k} = \sum_i m_i \mathbf{v}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} \quad \dots (9)$$

[using eqn. (10) of 15.1]

$$= \sum_i m_i \mathbf{v}_i \cdot \hat{n} = \hat{n} \cdot \sum_i m_i \mathbf{v}_i$$

which shows that p_k represents the component of total linear momentum along the direction of translation.

So we can say that the eqn. (7) i.e. $\dot{p}_k = Q_k$

represents the equation of motion for total linear momentum.

If $Q_k = 0$, $\dot{p}_k = 0$ i.e. $p_k = \text{constant}$.

This gives the conservation theorem for linear momentum and states that *if a given component of the total applied force vanishes, the corresponding component of the linear momentum is conserved.*

(iv) **Conservation Theorem for Angular Momentum.** Consider a conservative system so that the potential energy V is dependent on position only and the kinetic energy is independent of position, i.e.

$$\frac{\partial V}{\partial \dot{q}_k} = 0 \text{ and } \frac{\partial T}{\partial q_k} = 0. \quad \dots (10)$$

The Lagrangian equation for such a system is

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} = 0$$

i.e.

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) = \frac{\partial L}{\partial q_k} = \frac{\partial}{\partial q_k} (T - V) = - \frac{\partial V}{\partial q_k}$$

i.e.

$$\dot{p}_k = Q_k \quad \dots (11)$$

Now if we show that with rotation co-ordinate q_k , the generalised force Q_k is the component of total applied torque along the axis of rotation and the generalised momentum p_k is the component of total angular momentum along the same axis, then eqn. (11) will represent the equation of motion for the total angular momentum.

The generalised force Q_k is given by

$$Q_k = \sum_i \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k}$$

The magnitude of change in position co-ordinate $r_i(q_k)$ due to a change in rotation co-ordinate q_k is given by

$$\begin{aligned} |\delta \mathbf{r}_i| &= AB \delta q_k \\ &= r_i \sin \theta \delta q_k \\ \left| \frac{\partial \mathbf{r}_i}{\partial q_k} \right| &= r_i \sin \theta. \end{aligned} \quad \dots (12)$$

If $\hat{\mathbf{n}}$ is the unit vector along the axis of rotation, eqn. (12) may be written as

$$\frac{\partial \mathbf{r}_i}{\partial q_k} = \hat{\mathbf{n}} \times \mathbf{r}_i \quad \dots (13)$$

Now the generalised force can be expressed as

$$\begin{aligned} Q_k &= \sum_i \mathbf{F}_i \cdot \hat{\mathbf{n}} \times \mathbf{r}_i = \sum_i \hat{\mathbf{n}} \cdot (\mathbf{r}_i \times \mathbf{F}_i) \\ &= \sum_i \hat{\mathbf{n}} \cdot \boldsymbol{\tau}_i = \hat{\mathbf{n}} \cdot \sum_i \boldsymbol{\tau}_i = \hat{\mathbf{n}} \cdot \boldsymbol{\tau} \end{aligned} \quad \dots (14)$$

where $\boldsymbol{\tau} = \sum_i \boldsymbol{\tau}_i$ is the total torque. Eqn. (14) shows that Q_k is the component of the total torque along the axis of rotation.

Also

$$\begin{aligned} p_k &= \frac{\partial L}{\partial \dot{q}_k} = \frac{\partial T}{\partial \dot{q}_k} = \frac{\partial}{\partial \dot{q}_k} \left(\sum_i \frac{1}{2} m_i \dot{\mathbf{r}}_i^2 \right) = \sum m_i \mathbf{v}_i \cdot \frac{\partial \mathbf{r}_i}{\partial \dot{q}_k} \quad [\text{using eqn. (9)}] \\ &= \sum_i m_i \mathbf{v}_i \cdot \hat{\mathbf{n}} \times \mathbf{r}_i \quad \text{using (13)} \\ &= \sum_i \hat{\mathbf{n}} \cdot \mathbf{r}_i \times m_i \mathbf{v}_i \\ &= \hat{\mathbf{n}} \cdot \sum_i \mathbf{L}_i = \hat{\mathbf{n}} \cdot \mathbf{L} \end{aligned} \quad \dots (15)$$

where $\mathbf{L} = \sum_i \mathbf{L}_i$ is the total angular momentum. Eqn. (15) shows that p_k is the component of total angular momentum along the axis of rotation. So we can say that eqn. (11), i.e.

$$\dot{p}_k = Q_k, \quad \dots (16)$$

represents the equation of motion for the total angular momentum of the system.

If q_k is cyclic co-ordinate, then

$$\hat{\mathbf{n}} \cdot \boldsymbol{\tau} = Q_k = -\frac{\partial V}{\partial q_k} = \frac{\partial L}{\partial q_k} = 0.$$

Hence equation (16) yields

$$\dot{p}_k = 0,$$

i.e.

$$p_k = \hat{\mathbf{n}} \cdot \mathbf{L} = \text{constant}$$

Thus the conservation theorem for angular momentum states that if the rotation coordinate q_k is cyclic i.e. if the component of applied torque along the axis of rotation vanishes, then the component of total angular momentum along the axis of rotation is conserved.

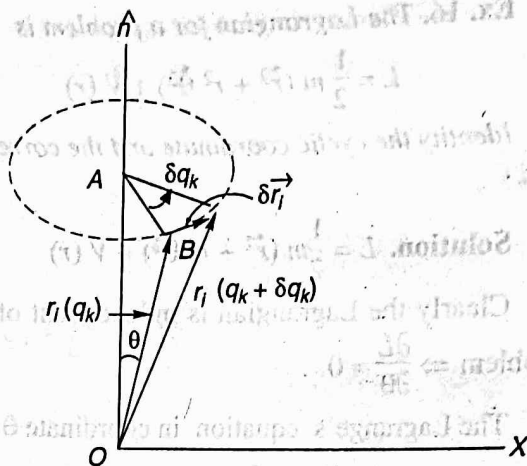


Fig. 15-10

Ex. 16. The Lagrangian for a problem is

$$L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) + V(r)$$

Identify the cyclic coordinate and the corresponding conservation law for the problem

(Meerut 1998)

Solution. $L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) + V(r)$

... (1)

Clearly the Lagrangian is independent of coordinate θ , so θ is the cyclic coordinate of the problem $\Rightarrow \frac{\partial L}{\partial \theta} = 0$

The Lagrange's equation in coordinate θ is

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = 0$$

As $\frac{\partial L}{\partial \theta} = 0 \Rightarrow \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) = 0$

$\Rightarrow \frac{\partial L}{\partial \dot{\theta}} = \text{constant}$

from (1) $\frac{\partial L}{\partial \dot{\theta}} = \frac{1}{2} m r^2 \times 2\dot{\theta} = mr^2\dot{\theta}$

i.e. $\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} = \text{constant}$

Thus angular momentum $mr^2\dot{\theta}$ is conserved.

15.4. The Hamiltonian Formulation

15.4. (a) Phase space.

In Lagrangian formulation there are $3n$ equations of motion of the second order for a system containing n particles in the absence of holonomic constraints. For general considerations it is more convenient to write $6n$ partial differential equations of the first order in place of $3n$ equations of the second order. For the purpose there must be $6n$ degrees of freedom or $6n$ dimensional space known as *phase space*—sometimes also called the Γ -space. In phase space momenta are also regarded as independent variables like space co-ordinates. A single particle in phase space is specified by six co-ordinates, 3 position co-ordinates and 3 momentum co-ordinates. This six dimensional space is sometimes called the μ -space. The Γ -space is a superposition of μ -spaces.

In brief phase space is a $6n$ -dimensional space formed of co-ordinates $q_1, q_2, \dots, q_{3n}, p_1, p_2, \dots, p_k, \dots, p_{3n}$.

15.4. (b) The Hamiltonian Function H

The Lagrangian, in general is the function of

$$q_1, q_2, \dots, q_k, \dots, \dot{q}_1, \dot{q}_2, \dots, \dot{q}_k, \dots, t, \text{ i.e.,}$$

$$L = L(q_1, q_2, \dots, q_k, \dots, \dot{q}_1, \dot{q}_2, \dots, \dot{q}_k, \dots, t) = L(q_k, \dot{q}_k, t)$$

$$\frac{dL}{dt} = \sum_k \frac{\partial L}{\partial q_k} \dot{q}_k + \sum_k \frac{\partial L}{\partial \dot{q}_k} \ddot{q}_k + \frac{\partial L}{\partial t}$$

... (1)

The Lagrangian equation is

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} = 0$$

or,

$$\frac{\partial L}{\partial q_k} = \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) \quad \dots (2)$$

With this substitution equation (1) becomes.

$$\frac{dL}{dt} = \sum_k \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) \dot{q}_k + \sum_k \frac{\partial L}{\partial \dot{q}_k} \ddot{q}_k + \frac{\partial L}{\partial t}$$

$$= \sum_k \frac{d}{dt} \left\{ \frac{\partial L}{\partial \dot{q}_k} \dot{q}_k \right\} + \frac{\partial L}{\partial t}$$

i.e.,

$$\frac{d}{dt} \left\{ L - \sum_k \frac{\partial L}{\partial \dot{q}_k} \dot{q}_k \right\} = \frac{\partial L}{\partial t} \quad \dots (3)$$

But

$$\frac{\partial L}{\partial \dot{q}_k} = p_k = \text{generalised momentum}$$

The equation (3) may be written as:

$$\frac{d}{dt} \left\{ L - \sum_k p_k \dot{q}_k \right\} = \frac{\partial L}{\partial t}$$

$$\frac{d}{dt} \left\{ \sum_k p_k \dot{q}_k - L \right\} = - \frac{\partial L}{\partial t} \quad \dots (4)$$

Now let us introduce a new function H known as **Hamiltonian function** defined as

$$H = \sum p_k \dot{q}_k - L(q, \dot{q}, t). \quad \dots (5)$$

It will be seen later on, that the function H is equal to the total energy in most of the physical problems. From equation (5) it is seen that Hamiltonian ' H ' is the function of p , q and t , i.e.,

$$H = H(p, q, t). \quad \dots (6)$$

Ex. 17. Obtain the Hamiltonian for an enharmonic oscillator, whose Lagrangian is

$$L(x, \dot{x}) = \frac{1}{2} \dot{x}^2 - \frac{1}{2} \omega^2 x^2 - \alpha x^3 \quad (\text{Kanpur 2000})$$

$$\text{Solution. Lagrangian } L(x, \dot{x}) = \frac{1}{2} \dot{x}^2 - \frac{1}{2} \omega^2 x^2 - \alpha x^3 \quad \dots (1)$$

$$p_x = \frac{\partial L}{\partial \dot{x}} = \dot{x} \quad \dots (2)$$

$$\text{Hamiltonian, } H = \sum_n p_k \dot{q}_k - L(q_k, \dot{q}_k, t)$$

In terms of coordinates x and $\dot{x}$, we have

$$H = p_x \dot{x} - L(x, \dot{x}, t)$$

Using (1) and (2), we get

$$\begin{aligned} H &= \dot{x}^2 - \left(\frac{1}{2} \dot{x}^2 - \frac{1}{2} \omega^2 x^2 - \alpha x^3 \right) \\ &= \frac{1}{2} \dot{x}^2 + \frac{1}{2} \omega^2 x^2 + \alpha x^3 \end{aligned}$$

This is required Hamiltonian.

15.4. (c) Hamilton's Equations

As seen above the Hamiltonian function H is function of p, q and t , i.e.,

$$H = H(p, q, t)$$

$$= H(p_1, p_2, \dots, p_k, \dots, q_1, q_2, \dots, q_k, \dots, t)$$

$$dH = \sum_k \frac{\partial H}{\partial p_k} dp_k + \sum_k \frac{\partial H}{\partial q_k} dq_k + \frac{\partial H}{\partial t} dt \quad \dots (1)$$

Also from equation (5) of section 15.4 b,

$$dH = \sum_k \dot{q}_k dp_k + \sum_k p_k d\dot{q}_k - dL, \quad \dots (2)$$

Since $L = L(q_1, q_2, \dots, q_k, \dots, \dot{q}_1, \dot{q}_2, \dots, \dot{q}_k, \dots, t)$,

$$\begin{aligned} dL &= \sum_k \frac{\partial L}{\partial q_k} dq_k + \sum_k \frac{\partial L}{\partial \dot{q}_k} d\dot{q}_k + \frac{\partial L}{\partial t} dt \\ &= \sum_k \frac{\partial L}{\partial q_k} dq_k + \sum_k p_k d\dot{q}_k + \frac{\partial L}{\partial t} dt \quad \dots (3) \end{aligned}$$

Substituting the value of dL from equation (3) in (2), we have

$$\begin{aligned} dH &= \sum_k \dot{q}_k dp_k + \sum_k p_k d\dot{q}_k - \sum_k \frac{\partial L}{\partial q_k} dq_k - \sum_k p_k d\dot{q}_k - \frac{\partial L}{\partial t} dt \\ &= \sum_k \dot{q}_k dp_k - \sum_k \frac{\partial L}{\partial t} dt \quad \dots (4) \end{aligned}$$

Comparing coefficients of dp_k, dq_k and dt in equations (1) and (4), are get

$$\left. \begin{aligned} \dot{q}_k &= \frac{\partial H}{\partial p_k} \\ -\dot{p}_k &= \frac{\partial H}{\partial q_k} \end{aligned} \right\} \quad \dots (5)$$

$$-\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t} \quad \dots (6)$$

Equations (5) are referred as *Hamilton's equations* or *Hamilton's canonical equations of motion*.

In the case of systems of n particles there are $6n$ partial differential equations of the first order in terms of rectangular co-ordinates in place of $3n$ Lagrangian equations of the second order.

15.4. (d) Physical Significance of the Hamiltonian Function

Hamiltonian function ' H ' is function of p, q and t , i.e. ... ,

$$H = H(p_1, p_2, p_k, \dots, q_1, q_2, \dots, q_k, \dots, t)$$

The total time derivative of H is

$$\begin{aligned} \frac{dH}{dt} &= \sum_k \frac{\partial H}{\partial q_k} \dot{q}_k + \sum_k \frac{\partial H}{\partial p_k} \dot{p}_k + \frac{\partial H}{\partial t} \\ &= - \sum_k \dot{p}_k \dot{q}_k + \sum_k \dot{q}_k \dot{p}_k + \frac{\partial H}{\partial t} \quad [\text{from equations (5) of § 15.4c}] \\ &= \frac{\partial H}{\partial t} \quad \dots (1) \end{aligned}$$

But according to equation (6) of 15.4c.

$$\frac{\partial H}{\partial t} = - \frac{\partial L}{\partial t}$$

Therefore

$$\frac{dH}{dt} = - \frac{\partial L}{\partial t} \quad \dots (2)$$

If L is not an explicit function of time, $\frac{\partial L}{\partial t} = 0$, thereby giving

$$\frac{dH}{dt} = 0,$$

i.e.,

$$H = \text{constant}$$

Thus we may state that if the *Lagrangian L* is not an explicit function of time, the *Hamiltonian H* is constant of motion.

For conservative systems the potential energy V does not depend upon generalised velocity, i.e.,

$$\frac{\partial V}{\partial \dot{q}_k} = 0. \quad \dots (5)$$

(1) Also we know

$$\begin{aligned} H &= \sum_k p_k \dot{q}_k - L = \sum_k \frac{\partial L}{\partial \dot{q}_k} \dot{q}_k - L \\ &= \sum_k \dot{q}_k \left\{ \frac{\partial}{\partial \dot{q}_k} (T - V) \right\} - L \quad (\text{since } L = T - V) \\ &= \sum_k \dot{q}_k \left\{ \frac{\partial T}{\partial \dot{q}_k} - \frac{\partial V}{\partial \dot{q}_k} - L \right\} = \sum_k \dot{q}_k \frac{\partial T}{\partial \dot{q}_k} - L \quad \left[\text{since } \frac{\partial V}{\partial \dot{q}_k} = 0 \right] \\ &= \sum_k \dot{q}_k \frac{\partial}{\partial \dot{q}_k} \left(\sum_i \frac{1}{2} m_i \dot{q}_i^2 \right) - L \quad (\text{since } T = \sum_i \frac{1}{2} m_i \dot{q}_i^2) \\ &= \sum_k m_k \dot{q}_k^2 - L \\ &= 2T - L \\ &= 2T - (T - V) = 2T - T + V \\ &= T + V \\ &= \text{K.E.} + \text{P.E.} \\ &= E = \text{total energy of the system.} \end{aligned}$$

Thus for conservative systems where the co-ordinate transformation is independent of time, the Hamiltonian function H represents the total energy of the system.

15.4. (e) Cyclic Co-ordinates and Routh's Procedure

It is to be noted that the Hamiltonian equations are specially used to solve the problems involving cyclic co-ordinates.

A co-ordinate q_k is cyclic if it does not appear in the Lagrangian,

i.e. for a cyclic coordinate q_k , $\frac{\partial L}{\partial q_k} = 0$

From Lagrange's equation, we have

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} = 0$$

or
$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) = \frac{\partial L}{\partial q_k}$$

or
$$\frac{d}{dt} (p_k) = \frac{\partial L}{\partial q_k}$$

[since $p_k = \frac{\partial L}{\partial \dot{q}_k}$ = generalised momentum corresponding to generalised co-ordinate q_k]

or
$$\dot{p}_k = \frac{\partial L}{\partial q_k}$$

Also
$$\dot{p}_k = - \frac{\partial H}{\partial q_k} \quad \text{[from eqn. (5) of § 15.4c]}$$

So
$$\dot{p}_k = \frac{\partial L}{\partial q_k} = - \frac{\partial H}{\partial q_k} \quad \dots (1)$$

If q_k is cyclic $\frac{\partial L}{\partial q_k} = 0$ therefore equation (1) implies $\frac{\partial H}{\partial q_k} = 0$.

Therefore, if any co-ordinate q_k is cyclic, $\frac{\partial H}{\partial q_k} = 0$.

So
$$\dot{p}_k = 0 \quad \text{[from eqn. (1)]}$$

or
$$p_k = \text{constant} = \alpha \text{ (say)} \quad \dots (2)$$

As $\frac{\partial H}{\partial q_k} = 0$, q_k is absent from Hamiltonian, hence the Hamiltonian may be expressed as

$$H = H(q_1, q_2, \dots, q_{k-1}, q_{k+1}, \dots, q_n, p_1, p_2, p_{k-1}, \alpha, p_{k+1}, \dots, p_n, t).$$

Also
$$\dot{q}_k = \frac{\partial H}{\partial p_k} \quad \text{[from eqn. (5) of § 15.4 c]}$$

$$= \frac{\partial H}{\partial \alpha} \quad \text{[from eqn. (2)]} \quad \dots (3)$$

To solve the problems involving cyclic and non-cyclic co-ordinates Routh devised a method by combining the Lagrangian and Hamiltonian procedures, being known as *Routh's procedure* after his name.

If $q_1, q_2, \dots, q_s$ are cyclic co-ordinates, then a new function termed as *Routhian* is defined by the relation

$$R(q_1, q_2, \dots, q_k, \dots, q_n, p_1, p_2, \dots, p_s, \dot{q}_1, \dots, \dot{q}_k, \dots, \dot{q}_n, t) = \sum_{k=1}^s p_k \dot{q}_k - L \quad \dots (4)$$

Here we have considered the problem involving n co-ordinates. The differential of R is

$$dR = \sum_{k=1}^s \dot{q}_k dp_k + \sum_{k=1}^s p_k d\dot{q}_k - \sum_{k=1}^n \frac{\partial L}{\partial \dot{q}_k} d\dot{q}_k - \sum_{k=1}^n \frac{\partial L}{\partial q_k} dq_k - \frac{\partial L}{\partial t} dt$$

$$\therefore dR = \sum_{k=1}^s \dot{q}_k dp_k + \sum_{k=1}^s \frac{\partial L}{\partial \dot{q}_k} d\dot{q}_k - \left\{ \sum_{k=1}^s \frac{\partial L}{\partial \dot{q}_k} d\dot{q}_k + \sum_{k=s+1}^n \frac{\partial L}{\partial \dot{q}_k} d\dot{q}_k \right\} - \sum_{k=1}^n \frac{\partial L}{\partial q_k} dq_k - \frac{\partial L}{\partial t} dt$$

(since $p_k = \frac{\partial L}{\partial \dot{q}_k}$)

$$\therefore dR = \sum_{k=1}^s \dot{q}_k dp_k - \sum_{k=s+1}^n \frac{\partial L}{\partial \dot{q}_k} d\dot{q}_k - \sum_{k=1}^n \frac{\partial L}{\partial q_k} dq_k - \frac{\partial L}{\partial t} dt \quad \dots (5)$$

which yields $\frac{\partial R}{\partial p_k} = \dot{q}_k, \frac{\partial R}{\partial q_k} = \dot{p}_k$ for $k = 1$ to s ... (6)

and $\frac{\partial R}{\partial \dot{q}_k} = -\frac{\partial L}{\partial \dot{q}_k}, \frac{\partial R}{\partial q_k} = -\frac{\partial L}{\partial q_k}$ for $k = s+1$ to n (7)

(since $\frac{\partial L}{\partial q_k} = \dot{p}_k$)

Equations (6) represent Hamilton's equations with R as Hamiltonian while equation (7) point out that the co-ordinates from $k = s+1$ to n form Lagrangian equations with R as the Lagrangian function, i.e.,

$$\frac{d}{dt} \left(\frac{\partial R}{\partial \dot{q}_k} \right) - \frac{\partial R}{\partial q_k} = 0 \text{ for } k = s+1 \text{ to } n \quad \dots (8)$$

A co-ordinate which does not appear in the Lagrangian, will be absent from the Routhian R . Here $q_1, q_2, \dots, q_s$ are cyclic, therefore they will not be involved in the Routhian R and the momenta $p_1, p_2, \dots, p_s$, conjugate to cyclic co-ordinates $q_1, q_2, \dots, q_s$ respectively, are constants and hence may be replaced by a set of constants $\alpha_1, \alpha_2, \dots, \alpha_s$ to be determined by boundary conditions.

Keeping in mind this fact the Routhian R may be stated to involve $(n-s)$ non-cyclic co-ordinates as variables and the generalised velocities corresponding to these $(n-s)$ co-ordinates. Thus R may be represented as

$$R = R(q_{s+1}, \dots, q_n, \dot{q}_{s+1}, \dots, \dot{q}_n, \alpha_1, \alpha_2, \dots, \alpha_s, t)$$

thereby indicating that the Lagrange's equations for non-cyclic co-ordinates can be solved as if the cyclic co-ordinates are absent and for cyclic co-ordinates we have Hamiltonian equations as usual.

Thus, according to Routhian procedure a problem involving cyclic and non-cyclic co-ordinates can be solved by solving Lagrangian equations for non-cyclic co-ordinates with R as the Lagrangian function and solving Hamiltonian equations for given cyclic co-ordinates with R as Hamiltonian function.

15.4 (f) Applications of Hamiltonian Equations

We have seen that dynamical problems can be solved easily using generalised co-ordinates and Lagrange's equations. In fact, there is hardly a mechanical problem where generalised co-ordinates are not applicable. For the solution of problems Hamiltonian equations are generally not so convenient as the Lagrangian equations and for their relation to more advanced mechanics, particularly to three fields; Celestial Mechanics, Statistical Mechanics and Quantum Mechanics. Here we shall consider some examples concerned with the dynamics of the particles to illustrate the applications of Hamiltonian equations.

Ex. 18. Explain the motion for a particle in central force field or Planetary Motion.

Or

Write down the Hamiltonian and Hamilton's equation for a particle in central force field in space. (Agra 1994)

Or

A particle moves in the XY-plane under the influence of a central force depending only on its distance from the origin.

(a) Set up the Hamiltonian for the system.

(b) Write Hamiltonian equations of motion.

(Meerut 1987)

Solution. If (r, θ) are the polar co-ordinates of the particle of mass m the Lagrangian L is $V(r)$ being potential energy due to central force.

$$L = T - V(r),$$

$$= \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) - V(r). \quad \dots (1)$$

Then have

$$\left. \begin{aligned} p_r &= \frac{\partial L}{\partial \dot{r}} = m \dot{r} \\ p_\theta &= \frac{\partial L}{\partial \dot{\theta}} = m r^2 \dot{\theta} \end{aligned} \right\} \quad \dots (2)$$

$$H = \sum p_k \dot{q}_k - L$$

$$= p_r \dot{r} + p_\theta \dot{\theta} - L$$

$$= (m \dot{r}) \dot{r} + (m r^2 \dot{\theta}) \dot{\theta} - L \quad \text{from (2)}$$

$$= m \dot{r}^2 + m r^2 \dot{\theta}^2 - \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) + V(r) \quad \text{from (1)}$$

$$= \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) + V(r) = T + V(r) \quad \dots (3)$$

$$= \text{Total energy.}$$

From (2) we have

$$\dot{r} = \frac{p_r}{m}, \quad \dot{\theta} = \frac{p_\theta}{m r^2}$$

Which these substitutions the Hamiltonian takes the form [See equation (3)]

$$\begin{aligned} H &= \frac{1}{2} m \left\{ \left(\frac{p_r}{m} \right)^2 + r^2 \left(\frac{p_\theta}{m r^2} \right)^2 \right\} + V(r) \\ &= \frac{1}{2m} \left(p_r^2 + \frac{p_\theta^2}{r^2} \right) + V(r). \quad \dots (4) \end{aligned}$$

This is the total energy expressed in terms of co-ordinates and momenta.

The Hamiltonian equations, corresponding to generalised co-ordinates q_k , are

$$\dot{q}_k = \frac{\partial H}{\partial p_k},$$

$$-\dot{p}_k = \frac{\partial H}{\partial q_k}.$$

In this case we have two co-ordinates r and θ ; therefore there will be four Hamilton's equations.

The two equations for $\dot{q}_k$ are :

$$\left. \begin{aligned} \dot{r} &= \frac{\partial H}{\partial p_r} = \frac{p_r}{m} \\ \dot{\theta} &= \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2} \end{aligned} \right\} \quad \begin{array}{l} \dots (5) \\ \text{from (4)} \end{array}$$

The two equations from $\dot{p}_k$ are :

$$\left. \begin{aligned} \dot{p}_r &= -\frac{\partial H}{\partial r} = \frac{p_\theta^2}{mr^3} - \frac{\partial V(r)}{\partial r} \\ \dot{p}_\theta &= -\frac{\partial H}{\partial \theta} = 0 \text{ i.e. angular momentum } p_\theta = \text{constant} \end{aligned} \right\} \quad \dots (6)$$

Ex. 19. (a) Derive Hamilton's equations in cylindrical coordinates

(b) For a particle of mass m moving on the inner surface of a frictionless vertical cone given by

$$x^2 + y^2 = z^2 \tan^2 \alpha$$

Obtain the Hamiltonian and the Hamilton's equations using cylindrical coordinates.

(Rohilkhand 2000)

Solution. (a) The transformation equations relating cartesian and cylindrical coordinates are

$$\left. \begin{aligned} x &= r \cos \theta \\ y &= r \sin \theta \\ z &= z \end{aligned} \right\}$$

$$\therefore \text{Kinetic energy } T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2 + \dot{z}^2)$$

$$\Rightarrow \dot{p}_r = \frac{\partial T}{\partial \dot{r}} = m\dot{r} \Rightarrow \dot{r} = \frac{p_r}{m}$$

$$p_\theta = \frac{\partial T}{\partial \dot{\theta}} = mr^2 \dot{\theta} \Rightarrow \dot{\theta} = \frac{p_\theta}{mr^2}$$

$$p_z = \frac{\partial T}{\partial \dot{z}} = m\dot{z} \Rightarrow \dot{z} = \frac{p_z}{m}$$

$$\text{Hamiltonian of system, } H = \frac{1}{2} m \left(p_r^2 + \frac{p_\theta^2}{r^2} + p_z^2 \right) + V$$

$\therefore$ Hamilton's equations are

$$\dot{p}_r = -\frac{\partial H}{\partial r} \Rightarrow \dot{p}_r = \frac{p_\theta^2}{mr^3} - \frac{\partial V}{\partial r}$$

$$\dot{r} = \frac{\partial H}{\partial p_r} \Rightarrow \dot{r} = \frac{p_r}{m}$$

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} \Rightarrow \dot{p}_\theta = -\frac{\partial V}{\partial \theta}$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} \Rightarrow \dot{\theta} = \frac{p_\theta}{mr^2}$$

$$\text{and } \dot{p}_z = -\frac{\partial H}{\partial z} \Rightarrow \dot{p}_z = -\frac{\partial V}{\partial z}$$

and $\dot{z} = \frac{\partial H}{\partial p_z} \Rightarrow \dot{z} = \frac{p_z}{m}$

(b) Given $x^2 + y^2 = z^2 \tan^2 \alpha$

As $x^2 + y^2 = r^2 \Rightarrow r^2 = z^2 \tan^2 \alpha$ or $r = z \tan \alpha$... (1)

Hamiltonian $H = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2 + \dot{z}^2) + V$

As r is related to z , Therefore there are only two generalized coordinates from (1) $z = r \cot \alpha$

$$\Rightarrow \dot{z} = \dot{r} \cot \alpha$$

$$H = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2 + \dot{r}^2 \cot^2 \alpha) + mgz$$

$$= \frac{1}{2} m (\dot{r}^2 \operatorname{cosec}^2 \alpha + r^2 \dot{\theta}^2) + mgr \cot \alpha$$

$$= \frac{1}{2m} \left(p_r^2 \operatorname{cosec}^2 \alpha + \frac{p_\theta^2}{r^2} \right) + mgr \cot \alpha$$

Hamilton's equations are

$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m} \operatorname{cosec}^2 \alpha$$

$$\dot{p}_r = -\frac{\partial H}{\partial r} = \frac{p_\theta}{mr^3} - mg \cot \alpha$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2} \text{ and } \dot{p}_\theta = -\frac{\partial H}{\partial \theta} = 0$$

15.5. Conservation Theorems and Symmetry Properties

The symmetry (i.e., homogeneity and isotropy) of space leads to the theorems of conservation of linear and angular momenta, while the symmetry of time leads to the conservation theorem of energy.

Homogeneity of space. Conservation of Linear Momentum. Consider a system of particles. Let the systems be given an infinitesimal translation, represented by a quantity ϵ . Consequently each particle of the system is displaced by the same amount ϵ . If $\mathbf{r}_i$ is the position vector of i th particle, then $\mathbf{r}_i$ changes to $\mathbf{r}_i'$ given by

$$\mathbf{r}_i' = \mathbf{r}_i + \epsilon, \text{ i.e., } \delta \mathbf{r}_i = \mathbf{r}_i' - \mathbf{r}_i = \epsilon \quad \dots (1)$$

As a result of this infinitesimal translation, the Hamiltonian of the system changes by an amount dH given by

$$\begin{aligned} dH &= \sum_i \frac{\partial H}{\partial \mathbf{r}_i} \cdot \delta \mathbf{r}_i = \sum_i \frac{\partial H}{\partial \mathbf{r}_i} \epsilon \\ &= \epsilon \sum_i \frac{\partial H}{\partial \mathbf{r}_i} \quad \dots (2) \end{aligned}$$

If space is homogeneous, the Hamiltonian of system will be invariant under translation, i.e., dH given by (2) must be zero. Further since ϵ in equation (2) is non-zero, we must have

$$\sum_i \frac{\partial H}{\partial \mathbf{r}_i} = 0 \quad \dots (3)$$

The Hamilton's canonical equation in coordinate $\mathbf{r}_i$ is

$$\dot{p}_i = -\frac{\partial H}{\partial \mathbf{r}_i}$$

∴ In view of this equation (3) gives

$$\sum_i \dot{p}_i = 0 \text{ or } \frac{d}{dt} \left(\sum_i p_i \right) = 0$$

Integrating $\sum_i p_i = \text{constant}$... (4)

That is the linear momentum of the system remains constant. Thus *the homogeneity of space leads to the law of conservation of linear momentum.*

Isotropy of space : Conservation of Angular Momentum : Consider a system of particles in the phase space. Consider i th particle of the system having spherical polar coordinates (r_i, θ_i, ϕ_i) . Let the system be given an infinitesimal rotation $\delta\phi$. Each particle of the system suffers this rotation. The resulting change $\delta\mathbf{r}_i$ in position $\mathbf{r}_i$ of the particle under consideration due to rotation is given by

$$\delta\mathbf{r}_i = \mathbf{r}_i \sin \theta_i \delta\phi$$

In vector form $\delta\mathbf{r}_i = \vec{\delta\phi} \times \mathbf{r}_i$... (5)

Similarly the change in momentum vector of the particle due to rotation is given by

$$\delta\mathbf{p}_i = \vec{\delta\phi} \times \mathbf{p}_i$$
 ... (6)

Since in phase space $\mathbf{r}_i$ and $\mathbf{p}_i$ behave similarly, the consequent change in Hamiltonian of the system is given by

$$\begin{aligned} dH &= \sum_i \frac{\partial H}{\partial \mathbf{r}_i} \cdot \delta\mathbf{r}_i + \sum_i \frac{\partial H}{\partial \mathbf{p}_i} \cdot \delta\mathbf{p}_i \\ &= \sum_i \frac{\partial H}{\partial \mathbf{r}_i} \cdot \vec{\delta\phi} \times \mathbf{r}_i + \sum_i \frac{\partial H}{\partial \mathbf{p}_i} \cdot \vec{\delta\phi} \times \mathbf{p}_i \end{aligned} \quad \dots (7)$$

Hamilton's canonical equations are

$$\frac{\partial H}{\partial \mathbf{r}_i} = -\dot{\mathbf{p}}_i \text{ and } \frac{\partial H}{\partial \mathbf{p}_i} = \dot{\mathbf{r}}_i \quad \dots (8)$$

Using (8), equation (7) becomes

$$\begin{aligned} dH &= \sum_i [-\dot{\mathbf{p}}_i \cdot \vec{\delta\phi} \times \mathbf{r}_i + \dot{\mathbf{r}}_i \cdot \vec{\delta\phi} \times \mathbf{p}_i] \\ &= - \sum_i (\mathbf{r}_i \times \dot{\mathbf{p}}_i + \dot{\mathbf{r}}_i \times \mathbf{p}_i) \cdot \vec{\delta\phi} \\ &= - \vec{\delta\phi} \cdot \frac{d}{dt} \sum_i (\mathbf{r}_i \times \mathbf{p}_i) \end{aligned} \quad \dots (9)$$

But $\sum_i \mathbf{r}_i \times \mathbf{p}_i = \mathbf{J}$, angular momentum of system.

$$\therefore dH = -\vec{\delta\phi} \cdot \frac{d\mathbf{J}}{dt}$$

If space is *isotropic*, the Hamiltonian of system does not change under rotation.

$$\therefore dH = 0$$

$$\therefore -\vec{\delta\phi} \cdot \frac{d\mathbf{J}}{dt} = 0 \quad \dots (10)$$

$$\text{As } \vec{\delta\phi} \text{ is arbitrary } \therefore \frac{d\mathbf{J}}{dt} = 0$$

or $\mathbf{J} = \text{constant}$.

Thus *the isotropy of space leads to the law of conservation of angular momentum.* ... (11)

Homogeneity of Time : Conservation of Energy

Consider a system of particles where time is homogeneous.

As $H = H(q_i, p_i, t)$

Taking total time derivative

$$\frac{dH}{dt} = \sum_i \frac{\partial H}{\partial q_i} \dot{q}_i + \sum_i \frac{\partial H}{\partial p_i} \dot{p}_i + \frac{\partial H}{\partial t} \quad \dots(12)$$

Hamilton's canonical equations are

$$\frac{\partial H}{\partial q_i} = -\dot{p}_i \text{ and } \frac{\partial H}{\partial p_i} = \dot{q}_i$$

Substituting these values in (12), we get

$$\frac{dH}{dt} = \sum_i (-\dot{p}_i \dot{q}_i + \dot{q}_i \dot{p}_i) + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t} \quad \dots(13)$$

The consequence of homogeneity of time is that Hamiltonian of system does not depend upon time explicitly, i.e., $\frac{\partial H}{\partial t} = 0$

$\therefore$ Equation (13) gives

$$\frac{dH}{dt} = 0$$

Integrating $H = \text{constant}$.

As for a conservative system the Hamiltonian represents the total energy of the system. Hence *the homogeneity of time leads to the law of conservation of total energy*.

15.6. Lagrangian and Hamiltonian of a Charged Particle in an Electromagnetic Field

The force experienced by a particle of charge q at rest in an electric field of intensity E is given by

$$\mathbf{F}_1 = q\mathbf{E}.$$

But, if the particle of charge q is in motion, the particle will experience an additional force which is linearly proportional to the velocity of charge. This additional force is called magnetic force and is given by

$$\mathbf{F}_2 = q(\mathbf{v} \times \mathbf{B}), \quad [\text{in S.I units}]$$

where $\mathbf{v}$ is the velocity of particle of charge q and $\mathbf{B}$ is the magnetic field intensity. The direction of $\mathbf{F}_2$ is normal to the plane containing $\mathbf{v}$ and $\mathbf{B}$.

Thus the total force on a uniformly moving particle of charge q is the sum of $\mathbf{F}_1$ and $\mathbf{F}_2$ i.e.,

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 = q\mathbf{E} + q(\mathbf{v} \times \mathbf{B}) \quad \dots(1)$$

The above equation is known as *Lorentz force equation* or Lorentz formula.

Maxwell's equations of electro-magnetic field in empty space or vacuum in S.I units are given by

$$\text{div } \mathbf{E} = \frac{\rho}{\epsilon_0} \quad \dots(a)$$

$$\text{div } \mathbf{B} = 0 \quad \dots(b)$$

$$\text{curl } \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \dots(c)$$

$$\text{curl } \mathbf{B} = \mu_0 \left(\epsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mathbf{J} \right) \quad \dots(d)$$

$\dots(2)$

where $\mathbf{j}$ and ρ are the current density and the density of electric charge respectively.

From equation (2b) we have $\text{div } \mathbf{B} = 0$, thereby giving a conclusion that there is a vector $\mathbf{A}$ such that

$$\mathbf{B} = \text{curl } \mathbf{A} \quad \dots(3)$$

Again from equation (2c), we have

(because $\text{div curl } \mathbf{A} = 0$).

$$\text{curl } \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = -\frac{\partial}{\partial t} (\text{curl } \mathbf{A}) = -\text{curl } \frac{\partial \mathbf{A}}{\partial t} \quad \text{from eqn. (3)}$$

$$\text{Therefore } \mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \text{grad } s \quad \dots(4)$$

which s is a scalar function.

[because $\text{curl grad } s = 0$]

The Lorentz force in terms of scalar potential s and vector potential $\mathbf{A}$ is given by

$$\begin{aligned} \mathbf{F} &= q \left[\mathbf{E} + (\mathbf{v} \times \mathbf{B}) \right] \\ &= q \left[-\frac{\partial \mathbf{A}}{\partial t} - \text{grad } s + (\mathbf{v} \times \text{curl } \mathbf{A}) \right] \quad \text{[from equations (1), (3) and (4)]} \\ &= q \left[-\frac{\partial \mathbf{A}}{\partial t} - \nabla s + \{ \mathbf{v} \times (\nabla \times \mathbf{A}) \} \right]. \quad \dots(5) \end{aligned}$$

Since $\mathbf{v} \times (\nabla \times \mathbf{A})$

$$\begin{aligned} &= \left[(\hat{\mathbf{i}} v_x + \hat{\mathbf{j}} v_y + \hat{\mathbf{k}} v_z) \times \left\{ \hat{\mathbf{i}} \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + \hat{\mathbf{j}} \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \hat{\mathbf{k}} \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \right\} \right] \\ &= \left[\hat{\mathbf{i}} \left\{ v_y \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) - v_z \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) \right\} + \hat{\mathbf{j}} \left\{ v_z \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) - v_x \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \right\} \right. \\ &\quad \left. + \hat{\mathbf{k}} \left\{ v_x \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) - v_y \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) \right\} \right] \\ &= \left[\left\{ \hat{\mathbf{i}} \left(v_y \frac{\partial A_y}{\partial x} + v_z \frac{\partial A_z}{\partial x} \right) + \hat{\mathbf{j}} \left(v_z \frac{\partial A_z}{\partial y} + v_x \frac{\partial A_x}{\partial y} \right) + \hat{\mathbf{k}} \left(v_x \frac{\partial A_x}{\partial z} + v_y \frac{\partial A_y}{\partial z} \right) \right. \right. \\ &\quad \left. \left. - \left\{ \hat{\mathbf{i}} \left(v_y \frac{\partial A_x}{\partial y} + v_z \frac{\partial A_x}{\partial z} \right) + \hat{\mathbf{j}} \left(v_z \frac{\partial A_y}{\partial z} + v_x \frac{\partial A_y}{\partial x} \right) + \hat{\mathbf{k}} \left(v_x \frac{\partial A_z}{\partial x} + v_y \frac{\partial A_z}{\partial y} \right) \right\} \right] \right]. \end{aligned}$$

Adding and subtracting $v_z \frac{\partial A_x}{\partial x}$, $v_y \frac{\partial A_y}{\partial y}$ and $v_z \frac{\partial A_z}{\partial z}$, we get

$$\begin{aligned} \mathbf{v} \times (\nabla \times \mathbf{A}) &= \left[\left\{ \hat{\mathbf{i}} \left(v_x \frac{\partial A_x}{\partial x} + v_y \frac{\partial A_y}{\partial y} + v_z \frac{\partial A_z}{\partial z} \right) \right. \right. \\ &\quad \left. \left. + \hat{\mathbf{j}} \left(v_x \frac{\partial A_x}{\partial y} + v_y \frac{\partial A_y}{\partial y} + v_z \frac{\partial A_z}{\partial y} \right) + \hat{\mathbf{k}} \left(v_x \frac{\partial A_x}{\partial z} + v_y \frac{\partial A_y}{\partial z} + v_z \frac{\partial A_z}{\partial z} \right) \right. \right. \\ &\quad \left. \left. - \left\{ \hat{\mathbf{j}} \left(v_x \frac{\partial A_x}{\partial x} + v_y \frac{\partial A_x}{\partial y} + v_z \frac{\partial A_x}{\partial z} \right) + \hat{\mathbf{i}} \left(v_x \frac{\partial A_y}{\partial x} + v_y \frac{\partial A_y}{\partial y} + v_z \frac{\partial A_y}{\partial z} \right) \right. \right. \right. \\ &\quad \left. \left. + \hat{\mathbf{k}} \left(v_x \frac{\partial A_z}{\partial x} + v_y \frac{\partial A_z}{\partial y} + v_z \frac{\partial A_z}{\partial z} \right) \right\} \right]. \end{aligned}$$

$$= \left[\left\{ \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) (A_x v_x + A_y v_y + A_z v_z) \right\} - \left\{ \left(v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z} \right) (\hat{i} A_x + \hat{j} A_y + \hat{k} A_z) \right\} \right] \quad (3)$$

$$(0 = \nabla(A \cdot v) - (\nabla \cdot v) A.$$

Also $A = A(x, y, z, t).$

Therefore total time derivative of A is given by

$$\begin{aligned} \frac{dA}{dt} &= \frac{\partial A}{\partial x} \dot{x} + \frac{\partial A}{\partial y} \dot{y} + \frac{\partial A}{\partial z} \dot{z} + \frac{\partial A}{\partial t} = \frac{\partial A}{\partial x} v_x + \frac{\partial A}{\partial y} v_y + \frac{\partial A}{\partial z} v_z + \frac{\partial A}{\partial t} \\ &= (\hat{i} v_x + \hat{j} v_y + \hat{k} v_z) \cdot \left(\hat{i} \frac{\partial A}{\partial x} + \hat{j} \frac{\partial A}{\partial y} + \hat{k} \frac{\partial A}{\partial z} \right) + \frac{\partial A}{\partial t} \\ &= v \cdot \nabla A + \frac{\partial A}{\partial t} \end{aligned} \quad \dots(7)$$

$$\text{or } v \cdot \nabla A = \frac{dA}{dt} - \frac{\partial A}{\partial t} \quad \dots(8)$$

Substituting the value of $v \cdot \nabla A$ from above equation in (6), we have

$$v \times (\nabla \times A) = \nabla(A \cdot v) - \frac{dA}{dt} + \frac{\partial A}{\partial t}.$$

With this substitution equation (5) becomes

$$\begin{aligned} F &= q \left[-\frac{\partial A}{\partial t} - \nabla s + \left\{ \nabla(A \cdot v) - \frac{dA}{dt} + \frac{\partial A}{\partial t} \right\} \right] \\ &= q \left[-\nabla s + \left\{ \nabla(A \cdot v) - \frac{dA}{dt} \right\} \right] \\ &= q \left[-\nabla(s - A \cdot v) - \frac{dA}{dt} \right]. \end{aligned} \quad \dots(9)$$

The x-component of the force is given by

$$\begin{aligned} F_x &= q \left[\left\{ -\nabla(s - A \cdot v) \right\}_x - \frac{dA_x}{dt} \right] \\ &= q \left[-\frac{\partial}{\partial x} (s - A \cdot v) - \frac{dA_x}{dt} \right] \quad \left[\text{since x-component of } \nabla \text{ is } \frac{\partial}{\partial x} \right] \\ \text{or } F_x &= q \left[-\frac{\partial}{\partial x} (s - A \cdot v) - \frac{d}{dt} \left\{ \frac{\partial}{\partial v_x} (A \cdot v) \right\} \right] \quad \dots(10) \\ &\quad \left[\text{since } \frac{\partial}{\partial v_x} (A \cdot v) = \frac{\partial}{\partial v_x} \{A_x v_x + A_y v_y + A_z v_z\} = A_x \right]. \end{aligned}$$

Since the scalar potential of qs is independent of velocity v_x i.e. $\frac{\partial(qs)}{\partial v_x} = 0$, the above equation is equivalent to

$$F_x = -\frac{\partial U}{\partial x} - \frac{d}{dt} \frac{\partial U}{\partial v_x}, \quad \dots(11)$$

where

$$U = qs - q(A \cdot v). \quad \dots(12)$$

It is clear that U is a function of x and v_x . In general U is function of q_k and $\dot{q}_k$ and is known as *generalised potential* or *velocity dependent potential*.

Equation (11) also represents in this case the generalised forces (even for non conservative systems) which are derivable from the generalised potential U .

The Lagrangian, defining the motion of the particle in an electromagnetic field, is given by

$$L = T - U = T - \left\{ qs - q (\mathbf{A} \cdot \mathbf{v}) \right\} = T - qs + q (\mathbf{A} \cdot \mathbf{v}) \quad \dots(13)$$

Here T is the K.E. of the particle and given by $T = \frac{1}{2} m v^2$.

With this substitution the **Lagrangian** is given by

$$L = \frac{1}{2} m v^2 - qs + q (\mathbf{A} \cdot \mathbf{v}) \quad \dots(14)$$

The canonical momenta are

$$p_k = \frac{\partial L}{\partial \dot{r}_k} = \frac{\partial L}{\partial v_k} = m v_k + q A_k \quad \dots(15)$$

from (2)

$$\begin{aligned} \text{So } p &= \sum_k p_k = \sum_k (m v_k + q A_k) \\ &= m \mathbf{v} + q \mathbf{A}. \end{aligned} \quad \dots(16)$$

Then the **Hamiltonian** H is

$$\begin{aligned} H &= \sum_k p_k \dot{q}_k - L \\ &= \sum_k p_k \dot{r}_k - L \\ &= \sum_k p_k v_k - L \\ &= \sum_k (m v_k + q A_k) v_k - \left[\frac{1}{2} m v^2 - qs + q (\mathbf{v} \cdot \mathbf{A}) \right] \\ &= \sum_k (m v_k^2 + q A_k v_k) - \frac{1}{2} m v^2 + qs - q (\mathbf{v} \cdot \mathbf{A}) \\ &= m v^2 + q (\mathbf{A} \cdot \mathbf{v}) - \frac{1}{2} m v^2 + qs - q (\mathbf{v} \cdot \mathbf{A}) \\ &= \frac{1}{2} m v^2 + qs \end{aligned} \quad \dots(17)$$

The Hamiltonian in terms of momenta is

$$H = \frac{1}{2} m \left(\frac{\mathbf{p}}{m} - \frac{q}{m} \mathbf{A} \right)^2 + qs \quad \dots(18)$$

The **Hamilton's canonical equations** are

$$\left[\begin{aligned} \dot{\mathbf{r}} &= \frac{\partial H}{\partial \mathbf{p}} \\ \dot{\mathbf{p}} &= -\nabla H \end{aligned} \right] \quad \dots(19)$$

and

From equation (5).

$$\frac{\partial H}{\partial \mathbf{p}} = \frac{1}{m} (\mathbf{p} - q \mathbf{A})$$

and $\nabla H = q \nabla s - q \nabla (\mathbf{v} \cdot \mathbf{A})$.

Substituting these values in (5), the Hamiltonian equations take the form

$$\left. \begin{aligned} \dot{\mathbf{r}} &= \mathbf{v} = \frac{1}{m} (\mathbf{p} - q \mathbf{A}) \\ \dot{\mathbf{p}} &= -q \nabla s + q \nabla (\mathbf{v} \cdot \mathbf{A}) \end{aligned} \right\} \quad \dots(20)$$

15.7. Variational Principles

15.7. (a) Derivation of Euler-Lagrange differential equation for one dependent and one independent variable. Variational principles deal with the problems in which the quantity to be minimized or maximized appear as an integral. In other words, it gives the necessary conditions that the quantity appearing as an integral has either a minimum or a maximum, i.e., stationary value.

Consider the simplest integral

$$J = \int_{x_1}^{x_2} f(y, y', x) dx \quad \dots(1)$$

Such that

$$y' = \frac{dy}{dx}$$

Here f is the known function of y , y' and x ; but the dependence of y on x is not specified initially, i.e. $y(x)$ is unknown.

The value of J is different along different parts connecting the points P and Q having coordinates (x_1, y_1) and (x_2, y_2) respectively. We have to choose the path of integration, i.e., $y(x)$, such that J has stationary value.

Consider two paths out of infinite number of possibilities as shown in figure 15.11 such that the difference between these two for the given value of x is the variation of y i.e., δy and may be described by introducing a new function $\eta(x)$ and ϵ to describe the arbitrary deformation of the path and the magnitude of variation respectively.

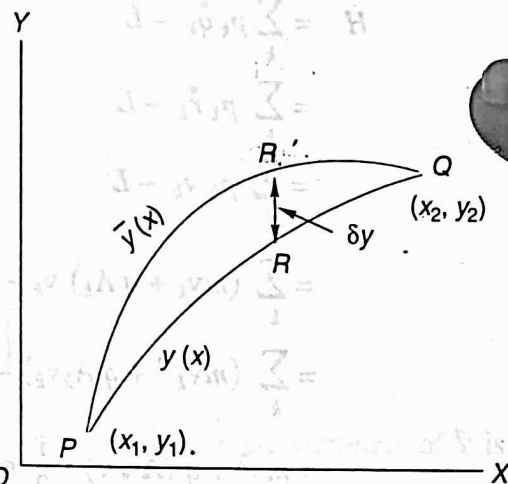


Fig. 15.11

The function $\eta(x)$ must satisfy the following two conditions :

(i) All varied paths must pass through the fixed points P and Q i.e.

$$\eta(x_1) = 0 \text{ and } \eta(x_2) = 0.$$

(2) $\eta(x)$ must be differentiable.

Let PRQ be the path along which J has stationary value and $PR'Q$ be the neighbouring path if $\bar{y}$ and $\bar{y}'$ are the values of y and y' along the varied path, then introducing $\eta(x)$ and ϵ , we have

$$\bar{y} = y + \delta y = y + \epsilon \eta(x)$$

$$\bar{y}' = y' + \delta y' = y' + \epsilon \eta'(x).$$

If the integral has stationary value along PRQ then

$$J(\epsilon) = \int_{x_1}^{x_2} f(x, y + \epsilon\eta, y' + \epsilon\eta') dx = \text{stationary value for } \epsilon = 0 \quad \dots(2)$$

$$\text{or } J(\epsilon) = \int_{x_1}^{x_2} \left[f(x, y, y') + \epsilon\eta \frac{\partial f}{\partial y} + \epsilon\eta' \frac{\partial f}{\partial y'} + \text{terms containing } \epsilon^2, \epsilon^3, \dots \right] dx. \quad \dots(3)$$

(expanding f by Taylor's series according to powers of ϵ).

Differentiating with respect to ϵ under the integral sign, we have

$$\frac{dJ}{d\epsilon} = \int_{x_1}^{x_2} \left(\eta \frac{\partial f}{\partial y} + \eta' \frac{\partial f}{\partial y'} + \text{terms containing } \epsilon, \epsilon^2 \text{ etc.} \right) dx.$$

Putting $\epsilon = 0$, we have the value of $\frac{dJ}{d\epsilon}$ over the path for which J has stationary value, i.e.,

$$\left(\frac{dJ}{d\epsilon} \right)_{\epsilon=0} = \int_{x_1}^{x_2} \left(\eta \frac{\partial f}{\partial y} + \eta' \frac{\partial f}{\partial y'} \right) dx. \quad \dots(4)$$

The condition for the path along which J has stationary value is

$$\left(\frac{dJ}{d\epsilon} \right)_{\epsilon=0} = 0$$

Thus the condition for an extremum is

$$\int_{x_1}^{x_2} \left(\eta \frac{\partial f}{\partial y} + \eta' \frac{\partial f}{\partial y'} \right) dx = 0 \quad \dots(5)$$

Integrating the second term by parts, i.e.,

$$\int_{x_1}^{x_2} \eta' \frac{\partial f}{\partial y'} dx = \left[\eta \frac{\partial f}{\partial y'} \right]_{x_1}^{x_2} - \int_{x_1}^{x_2} \eta \frac{d}{dx} \frac{\partial f}{\partial y'} dx = - \int_{x_1}^{x_2} \eta \frac{d}{dx} \frac{\partial f}{\partial y'} dx, \quad [\text{since } \eta(x_1) = \eta(x_2) = 0.]$$

With this substitution equation (5) becomes

$$\int_{x_1}^{x_2} \eta \frac{\partial f}{\partial y} dx - \int_{x_1}^{x_2} \eta \frac{d}{dx} \frac{\partial f}{\partial y'} dx = 0$$

$$\text{or } \int_{x_1}^{x_2} \eta \left\{ \frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} \right\} dx = 0. \quad \dots(6)$$

Since $\eta(x)$ is perfectly arbitrary, therefore the condition for the existence of an extremum can be satisfied if and only if the bracketted expression in the integral is identically zero, i.e.,

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} = 0$$

$$\text{or } \frac{d}{dx} \frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y} = 0$$

$$\text{or } \frac{d}{dx} \frac{\partial}{\partial y'} f(y, y', x) - \frac{\partial}{\partial x} f(y, y', x) = 0. \quad \dots(7)$$

This is known as *Euler Lagrange Differential equation*.

Thus the condition that the integral

$$J = \int_{x_1}^{x_2} f(x, y', x) dx$$

has an extremum value, is given by

$$\frac{d}{dx} \frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y} = 0. \quad \dots(8)$$

15.7 (b) Derivation of Euler-Lagrange differential equation for several dependent variables

Let us now consider that the integrand f occurring in the integral to be minimized or maximized is a function of one independent variable x and several dependent variables $y_1(x)$, $y_2(x)$, $y_3(x)$,.....

We want to find the functions $y_1(x)$, $y_2(x)$, $y_3(x)$,.....so that the integral

$$J = \int_{x_1}^{x_2} f[y_1, y_2, y_3, \dots, y_k, \dots, y_1', y_2', y_3', \dots, y_k', \dots, x] dx \quad \dots(1)$$

may be stationary.

Consider two paths out of infinite number of possibilities as shown in fig. 15.11, such that the difference between them may be described by introducing new functions $\eta_k(x)$ and ϵ .

The functions $\eta_k(x_2)$ must satisfy the following two conditions :

(1) All neighbouring paths must pass through the fixed points P and Q , i.e.

$$\eta_k(x_1) = \eta_k(x_2) = 0.$$

(2) $\eta_k(x)$ must be differentiable.

Let PRQ be the path along which J has stationary value and let $PR'Q$ be the neighbouring path. If $\bar{y}_k$ and $\bar{y}_k'$ are the values of y_k' and y_k along the varied path, then introducing $\eta_k(x)$ and ϵ , we have

$$\bar{y}_1 = y_1 + \delta y_1 = y_1 + \epsilon \eta_1(x),$$

$$\bar{y}_2 = y_2 + \delta y_2 = y_2 + \epsilon \eta_2(x),$$

$$\dots \quad \dots \quad \dots \quad \dots$$

$$\bar{y}_k = y_k + \delta y_k = y_k + \epsilon \eta_k(x).$$

If the integral has stationary value along PRQ , then

$$J(\epsilon) = \int_{x_1}^{x_2} f[y_1 + \epsilon \eta_1, y_2 + \epsilon \eta_2, \dots, y_k + \epsilon \eta_k, \dots, y_1' + \epsilon \eta_1', y_2' + \epsilon \eta_2', \dots, y_k' + \epsilon \eta_k', \dots, x] dx$$

$$= \text{stationary for } \epsilon = 0. \quad \dots(2)$$

Expanding above equation by Taylor's theorem and differentiating w.r.t. ϵ and then setting $\epsilon = 0$, we get

$$\left(\frac{dJ}{d\epsilon} \right)_{\epsilon=0} = \int_{x_1}^{x_2} \sum_k \left\{ \eta_k \frac{\partial f}{\partial y_k} + \eta_k' \frac{\partial f}{\partial y_k'} \right\} dx, \quad \dots(3)$$

The condition for the path along which J has stationary value is

$$\left(\frac{dJ}{d\varepsilon}\right)_{\varepsilon=0} = 0.$$

With this condition for J to be extremum, we have from eq. (3),

$$\int_{x_1}^{x_2} \sum_k \left\{ \eta_k \frac{\partial f}{\partial y_k} + \eta'_k \frac{\partial f}{\partial y'_k} \right\} dx = 0. \quad \dots(4)$$

Integrating the second term by parts, we get

$$\begin{aligned} \int_{x_1}^{x_2} \eta'_k \frac{\partial f}{\partial y'_k} dx &= \left\{ \eta_k \frac{\partial f}{\partial y'_k} \right\}_{x_1}^{x_2} - \int_{x_1}^{x_2} \eta_k \frac{d}{dx} \left(\frac{\partial f}{\partial y'_k} \right) dx. \\ &= - \int_{x_1}^{x_2} \eta_k \frac{d}{dx} \left(\frac{\partial f}{\partial y'_k} \right) dx \quad \text{since } \eta_k(x_1) = \eta_k(x_2) = 0. \end{aligned}$$

With this substitution equation (4) becomes

$$\begin{aligned} \int_{x_1}^{x_2} \sum_k \left\{ \eta_k \frac{\partial f}{\partial y_k} - \eta'_k \frac{d}{dx} \left(\frac{\partial f}{\partial y'_k} \right) \right\} dx &= 0 \\ \int_{x_1}^{x_2} \sum_k \eta_k \left\{ \frac{\partial f}{\partial y_k} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'_k} \right) \right\} dx &= 0. \quad \dots(5) \end{aligned}$$

Since η_k 's are perfectly arbitrary and independent of one another, each of the terms in equation (5) must vanish independently; we have

$$\begin{aligned} \frac{\partial f}{\partial y_k} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'_k} \right) &= 0 \quad \text{where } k = 1, 2, 3, \dots \\ \text{or} \quad \frac{d}{dx} \left(\frac{\partial f}{\partial y'_k} \right) - \frac{\partial f}{\partial y_k} &= 0. \quad \dots(6) \end{aligned}$$

Equation (6) represents a whole set of Euler Lagrange Equations, each of which must be satisfied for an extreme value.

SOLVED EXAMPLES

Ex. 20. Prove that if f does not depend on x explicitly, then $f - f' \frac{\partial f}{\partial y'} = \text{constant}$.

Solution. The Euler-Lagrange differential equation is

$$\frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) - \frac{\partial f}{\partial y} = 0.$$

Multiplying above equation by y' and adding and subtracting the expression $y'' \frac{\partial f}{\partial y'}$

(where $y'' = \frac{\partial y'}{\partial x}$ and $y' = \frac{\partial y}{\partial x}$), we get

$$y' \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) - y' \frac{\partial f}{\partial y} + y'' \frac{\partial f}{\partial y'} - y'' \frac{\partial f}{\partial y'} = 0$$

$$\begin{aligned}
 \text{or} \quad & y' \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) + y'' \frac{\partial f}{\partial y'} - y'' \frac{\partial f}{\partial y'} - y' \frac{\partial f}{\partial y} = 0 \\
 \text{or} \quad & \frac{d}{dx} \left(y' \frac{\partial f}{\partial y'} \right) - y'' \frac{\partial f}{\partial y'} - y' \frac{\partial f}{\partial y} - \frac{\partial f}{\partial x} + \frac{\partial f}{\partial x} = 0 \quad \left(\text{adding and subtracting } \frac{\partial f}{\partial x} \right) \\
 \text{or} \quad & \frac{d}{dx} \left(y' \frac{\partial f}{\partial y'} \right) - \left[y'' \frac{\partial f}{\partial y'} + y' \frac{\partial f}{\partial y} + \frac{\partial f}{\partial x} \right] + \frac{\partial f}{\partial x} = 0 \\
 \text{or} \quad & \frac{d}{dx} \left(y' \frac{\partial f}{\partial y'} \right) - \frac{df}{dx} + \frac{\partial f}{\partial x} = 0 \quad \{ \text{because } f = f(y, y', x) \} \\
 \text{or} \quad & \frac{d}{dx} \left[y' \frac{\partial f}{\partial y'} - f \right] + \frac{\partial f}{\partial x} = 0 \quad \dots(1)
 \end{aligned}$$

If f does not depend upon x explicitly, then $\frac{\partial f}{\partial x} = 0$ and hence we must have

$$\begin{aligned}
 & \frac{d}{dx} \left[y' \frac{\partial f}{\partial y'} - f \right] = 0 \\
 \text{or} \quad & y' \frac{\partial f}{\partial y'} - f = \text{constant} \\
 \text{i.e.} \quad & f - y' \frac{\partial f}{\partial y'} = \text{constant} \quad \dots(2)
 \end{aligned}$$

Geodesics

Ex. 21. Prove that the shortest distance between two points in a plane is a straight line.

(Kumaon 2004, 1999, Garhwal 2003, 1991, Meerut 2004, 1987; Rohilkhand, 2006, 2001)

Solution. An element of distance between two points in XY-plane is given by

$$ds^2 = dx^2 + dy^2$$

$$\text{or} \quad ds = (dx^2 + dy^2)^{1/2} = \left[1 + \left(\frac{dy}{dx} \right)^2 \right]^{1/2} dx$$

The total distance between two points having co-ordinates (x_1, y_1) and (x_2, y_2) is given by

$$J = \int_{x_1, y_1}^{x_2, y_2} ds = \int_{x_1}^{x_2} \left[1 + \left(\frac{dy}{dx} \right)^2 \right]^{1/2} dx$$

Comparing equations (1) of (15.5 a) and (1), we get

$$f(y, y', x) = \left[1 + \left(\frac{dy}{dx} \right)^2 \right]^{1/2} = \left[1 + y'^2 \right]^{1/2} \quad \text{where } y' = \frac{dy}{dx} \quad \dots(1)$$

If J is to be minimum, equation (8) of § (15.5) with $f = (1 + y'^2)^{1/2}$ must be satisfied, i.e., we must have

$$\begin{aligned}
 & \frac{d}{dx} \left[\frac{d}{dy'} (1 + y'^2)^{1/2} \right] - \frac{\partial}{\partial y} (1 + y'^2)^{1/2} = 0 \\
 \text{or} \quad & \frac{d}{dx} \left[\frac{y'}{(1 + y'^2)^{1/2}} \right] = 0 \\
 \text{or} \quad & \frac{y'}{(1 + y'^2)^{1/2}} = c, \quad \text{where } c \text{ is constant} \\
 \text{or} \quad & y' = c (1 + y'^2)^{1/2}
 \end{aligned}$$

or

$$y'^2 = c^2 (1 + y'^2)$$

or

$$y'^2 (1 - c^2) = c^2$$

or

$$y' = \frac{c}{\sqrt{1 - c^2}}$$

or

$$y = a,$$

$$\text{where } a = \frac{c}{\sqrt{1 - c^2}} = \text{constant}$$

or

$$\frac{dy}{dx} = a$$

$$\text{since } y' = \frac{dy}{dx}$$

or

$$dy = a dx.$$

Integrating, we get

$$y = ax + b,$$

where b is constant of integration,

which is the equation of a straight line. Hence Euler's equation predicts that the shortest distance between two fixed points in a plane is a straight line.

Ex. 22. Show that the sphere is the solid figure of revolution which has maximum volume for the given surface area. (Rohilkhand 1991, 84, 82)

Solution. The volume of sphere may be supposed to be formed of a large number of discs while the surface area may be supposed to be formed of a large number of rings.

The surface area of sphere A

$$\begin{aligned} &= 2\pi \int y ds = 2\pi \int y (dx^2 + dy^2)^{1/2} \\ &= 2\pi \int y \left[1 + \left(\frac{dy}{dx} \right)^2 \right]^{1/2} dx \\ &= 2\pi \int y [1 + y'^2]^{1/2} dx \end{aligned} \quad \text{where } y' = (dy/dx).$$

The volume of sphere,

$$V = \pi \int y^2 dx. \quad \dots(2)$$

Combining (1) and (2), we may write

$$\pi \int y^2 dx + \lambda 2\pi \int y (1 + y'^2)^{1/2} dx = \text{constant}.$$

or

$$\int [y^2 + \lambda y (1 + y'^2)^{1/2}] dx = \text{constant}$$

or

$$y^2 + \lambda y (1 + y'^2)^{1/2} = \text{extremum (say)}. \quad \dots(3)$$

As f does not depend on x explicitly, we have from the result of Ex. 19.

$$\left(y' \frac{\partial f}{\partial y'} - f \right) = \text{constant}. \quad \dots(4)$$

We have

$$f = y^2 + \lambda y (1 + y'^2)^{1/2}$$

$$\therefore \frac{\partial f}{\partial y'} = 0 + \lambda y \cdot \frac{1}{2} (1 + y'^2)^{-1/2} \times 2y' = \frac{\lambda y y'}{(1 + y'^2)^{1/2}}$$

Therefore equation (4) gives

$$\left[y' \cdot \frac{\lambda y y'}{(1 + y'^2)^{1/2}} - y^2 - \lambda y (1 + y'^2)^{1/2} \right] = \text{constant } C. \quad \dots(5)$$

But $y = 0$ at $x = 0$ (at origin), this gives constant C in (5) = 0.

$$\frac{\lambda y y'^2}{(1 + y'^2)^{1/2}} - y^2 - \lambda y (1 + y'^2)^{1/2} = 0$$

$$\text{or } \frac{\lambda y y'^2 - y^2 (1 + y'^2)^{1/2} - \lambda y (1 + y'^2)^{1/2}}{(1 + y'^2)^{1/2}} = 0$$

$$\text{or } \lambda y y'^2 - y^2 (1 + y'^2)^{1/2} - \lambda y (1 + y'^2)^{1/2} = 0$$

$$\text{or } -y^2 (1 + y'^2)^{1/2} - \lambda y = 0$$

$$\text{This gives } y(1 + y'^2)^{1/2} + \lambda = 0$$

$$\text{or } y(1 + y'^2)^{1/2} = -\lambda.$$

$$\text{Squaring, we get } y^2 (1 + y'^2) = \lambda^2.$$

Solving for y' , we get

$$y' = \frac{dy}{dx} = \frac{\sqrt{(\lambda^2 - y^2)}}{y}.$$

Integrating we obtain within limits x_0 to x and y_0 to y

$$\text{i.e., } -\sqrt{(\lambda^2 - y^2)} = x - x_0, \\ (x - x_0)^2 + y^2 = \lambda^2.$$

This equation represents a sphere with centre at x_0 on X-axis and radius λ . Hence we say that for the values of area A and volume V of sphere, f is extremum.

Thus the sphere is the solid figure of revolution, which has maximum volume for given surface area.

Ex. 23. Prove that the geodesics of a spherical surface are great circles i.e., the circles whose centres lie at the centre of the sphere. (Rohilkhand 1999)

Or

Prove that the shortest distance between two points on the surface of the sphere is the arc of the great circle connecting them. (Rohilkhand 1989, 86)

Solution. The element of distance ds , on the surface of sphere of radius r , in spherical co-ordinates, is given by

$$ds^2 = r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \quad \text{since } r \text{ is constant.}$$

$$\text{or } ds = r (d\theta^2 + \sin^2 \theta d\phi^2)^{1/2}.$$

The total distance between two points having co-ordinates $(r_1, \theta_1, \phi_1), (r_2, \theta_2, \phi_2)$ is given by

$$J = \int_{(\theta_1, \phi_1)}^{(\theta_2, \phi_2)} ds = \int_{\theta_1}^{\theta_2} r \left[1 + \sin^2 \theta \left(\frac{d\phi}{d\theta} \right)^2 \right]^{1/2} d\theta. \quad \dots(1)$$

Comparing equations (1) of § (15.5a) and (1), we get

$$f = r \left\{ 1 + \sin^2 \theta \left(\frac{d\phi}{d\theta} \right)^2 \right\}^{1/2}$$

If J is to be minimum or maximum, equation (8), of § (15.5 a) with

$$f = r \left\{ 1 + \sin^2 \theta \left(\frac{d\phi}{d\theta} \right)^2 \right\}^{1/2} \text{ must be satisfied, i.e., we must have}$$

$$\frac{d}{d\phi} \left[\frac{\partial}{\partial \phi} \left\{ 1 + \sin^2 \theta \left(\frac{d\phi}{d\theta} \right)^2 \right\}^{1/2} \right] - \frac{\partial}{\partial \theta} \left[r \left\{ 1 + \sin^2 \theta \left(\frac{d\phi}{d\theta} \right)^2 \right\}^{1/2} \right] = 0$$

$$\text{where } \phi' = \frac{\partial \phi}{\partial \theta}$$

$$\text{or } \frac{d}{d\theta} \left[r \left\{ \frac{\partial}{\partial \phi'} (1 + \sin^2 \theta \phi'^2)^{1/2} \right\} \right] = 0$$

$$\text{or } \frac{d}{d\theta} \left[\frac{2r \sin^2 \theta \phi'}{2(1 + \sin^2 \theta \phi'^2)^{1/2}} \right] = 0$$

$$\text{or } \frac{d}{d\theta} \left[\frac{\sin^2 \theta \phi'}{(1 + \sin^2 \theta \phi'^2)^{1/2}} \right] = 0, \quad \text{since } r \neq 0.$$

Integrating, we get $\frac{\sin^2 \theta \phi'}{(1 + \sin^2 \theta \phi'^2)^{1/2}} = c$, where c is any constant, which gives

$$\phi' = \frac{c \operatorname{cosec}^2 \theta}{(1 - c^2 - c^2 \cot^2 \theta)^{1/2}}.$$

Integrating again, we get

$$\phi = \alpha - \sin^{-1}(k \cot \theta), \quad \dots(2)$$

where α and k are new constants.

Transforming spherical polar co-ordinates into cartesian co-ordinates using transformations equation which are

$$\begin{aligned} x &= r \sin \theta \cos \phi, \\ y &= r \sin \theta \sin \phi, \\ z &= r \cos \theta, \end{aligned}$$

$$\text{we have } rk \cot \theta = r \sin(\alpha - \phi)$$

$$\text{or } kr \cos \theta = r \sin(\alpha - \phi) \sin \theta$$

$$zk = r [\sin \alpha \cos \phi - \cos \alpha \sin \phi] \sin \theta$$

$$\text{or } zk = x \sin \alpha - y \cos \alpha \quad \dots(3)$$

This equation represents a plane passing through the origin and hence cutting the surface of the sphere in great circle thereby indicating that the shortest or longest distance between two points on the surface of the sphere is the arc of the circle whose centre lies at the centre of the sphere.

Brachistochrone Problem

Ex. 24. show that the path followed by a particle in sliding from one point to another in the absence of friction in the shortest time is a cycloid. (Meerut 1986, Rohilkhand 1988)

Solution. The problem of finding the path followed by a particle in sliding from one point to another in the absence of friction in the shortest (brachistos) time (chronos) is called the problem of Brachistochrone.

Let the particle of mass m slide freely along a curve from point (x_1, y_1) to the origin under the influence of gravity. Let y -axis be vertically upwards and v the velocity of the particle at any point of its path. Then the time of descent is given by

$$J = \int_0^{x_1, y_1} \frac{ds}{v}$$

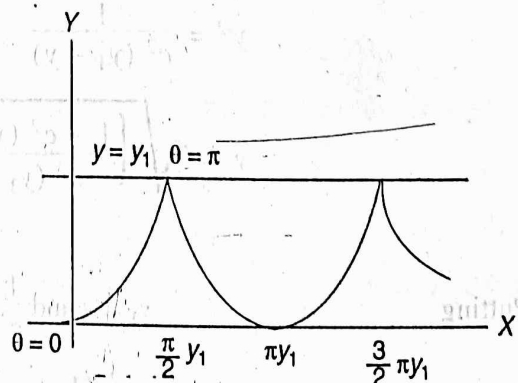


Fig. 15.12

From the principle of conservation of energy, we have

$$\frac{1}{2}mv^2 = mg(y_1 - y)$$

or

$$v = \sqrt{2g(y_1 - y)}$$

 $\therefore$

$$J = \int_0^{x_1, y_1} \frac{(dx^2 + dy^2)^{1/2}}{\sqrt{2g(y_1 - y)}} = \int_0^{x_1} \frac{[1 + (dy/dx)^2]^{1/2}}{\sqrt{2g(y_1 - y)}} dx$$

$$= \int_0^{x_1} \left[\frac{1 + y'^2}{2g(y_1 - y)} \right]^{1/2} dx \quad \text{since } y' = \frac{dy}{dx}$$

Comparing eqn. (1) of (§ 15.5 a) and (1), $J = \int f(y, y', x) dx = \text{extremum}$, we get

$$f = \left[\frac{1 + y'^2}{2g(y_1 - y)} \right]^{1/2}$$

$$\text{Therefore, } \frac{\partial f}{\partial y'} = \frac{1}{\sqrt{2g(y_1 - y)}} \cdot \frac{1}{2} (1 + y'^2)^{-1/2} \cdot 2y'$$

$$= \frac{y'}{\sqrt{2g(y_1 - y)} \sqrt{1 + y'^2}}$$

and

$$\frac{\partial f}{\partial x} = 0.$$

If J is to be minimized, we must have from equation (2) of Ex. 19, $\left[\text{i.e. } f - y' \frac{\partial f}{\partial y'} = c \right]$

$$\left[\frac{1 + y'^2}{2g(y_1 - y)} \right]^{1/2} - \frac{y'^2}{\sqrt{2g(y_1 - y)} \sqrt{1 + y'^2}} = c$$

$$\text{or } \frac{1 + y'^2 - y'^2}{[(y_1 - y)(1 + y'^2)]^{1/2}} = c \quad \text{or } \frac{1}{[(y_1 - y)(1 + y'^2)]^{1/2}} = c$$

Squaring, we get

$$\frac{1}{(y_1 - y)(1 + y'^2)} = c^2$$

or

$$c^2(y_1 - y)(1 + y'^2) = 1$$

or

$$1 + y'^2 = \frac{1}{c^2(y_1 - y)}$$

or

$$y'^2 = \frac{1}{c^2(y_1 - y)} - 1$$

or

$$y' = \sqrt{\left\{ \frac{1 - c^2(y_1 - y)}{c^2(y_1 - y)} \right\}} = \left[\frac{1}{c^2} - y_1 + y \right]^{1/2} \quad (2)$$

Putting

$$y_1 = a \text{ and } \frac{1}{c^2} - y_1 = b$$

or

$$y' = \left(\frac{b + y}{a - y} \right)^{1/2} \quad (4)$$

Introducing a new parameter θ , such that

$$y = \frac{a-b}{2} - \frac{a+b}{2} \cos \theta, \quad (5)$$

so that

$$dy = \frac{a+b}{2} \sin \theta d\theta. \quad (6)$$

From equation (4), we have

$$dy = \left(\frac{b+y}{a-y} \right)^{1/2} dx.$$

$$\text{or } \frac{a+b}{2} \sin \theta d\theta = \left[\frac{\left\{ b + \frac{a-b}{2} - \frac{a+b}{2} \cos \theta \right\}}{\left\{ a - \left(\frac{a-b}{2} - \frac{a+b}{2} \cos \theta \right) \right\}} \right]^{1/2} dx$$

$$\text{or } \frac{a+b}{2} \sin \theta d\theta = \left[\frac{(a+b)(1-\cos \theta)}{(a+b)(1+\cos \theta)} \right]^{1/2} dx$$

$$= \left[\frac{1-\cos \theta}{1+\cos \theta} \right]^{1/2} dx$$

$$\text{or } dx = \frac{a+b}{2} \left[\frac{1+\cos \theta}{1-\cos \theta} \right]^{1/2} \sin \theta d\theta$$

$$= \frac{a+b}{2} \left[\frac{2 \cos^2 \frac{\theta}{2}}{2 \sin^2 \frac{\theta}{2}} \right] 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} d\theta$$

$$= \frac{a+b}{2} 2 \cos^2 \frac{\theta}{2} d\theta$$

$$\text{or } dx = \frac{a+b}{2} (1+\cos \theta) d\theta.$$

$$\text{Integrating, we get } x + d = \frac{a+b}{2} (\theta + \sin \theta). \quad (7)$$

where d is a constant of integration.

$$\text{Equation (5) gives } y + b = \frac{a+b}{2} (1 - \cos \theta). \quad (8)$$

Equations (7) and (8) are the familiar equations of a cycloid. The constants b can be determined merely by remembering that cycloid passes through the origin and the point (x_1, y_1) while the constant a is fixed by suitable choice of y_1 , an initial condition of the problem.

Let us now consider a special case. When $x_1 = \frac{\pi}{2} y_1$, $b = d = 0$, then we have from (7) and (8)

$$\left. \begin{aligned} x &= \frac{y_1}{2} (\theta + \sin \theta), \\ y &= \frac{y_1}{2} (1 - \cos \theta). \end{aligned} \right\} \quad (9)$$

These are the equation of a cycloid described by the point on the circumference of a circle of radius $\frac{y_1}{2}$ rolling along the line $y = y_1$ upside down as shown in fig. 15-10.

Soap Film Problem

Ex. 25. A surface is generated by revolving the curve $y(x)$ along the axis of X . Apply variational principle to choose the curve $y(x)$, which passes through two fixed points (x_1, y_1) , so that the area of the resulting surface may be minimum. (Kumaon 1998, Meerut 1991)

OR

Show that the curve passing through two fixed end points with minimum surface area of revolution is a catenary. (Rohilkhand 2000)

Solution. Let the function $y(x)$ be revolved about x -axis so that the surface is generated. Consider an element of the surface in the neighbourhood of any P having co-ordinates (x, y) , as shown in fig. 15-13, having area $= 2\pi y ds$.

$$= 2\pi y (dx^2 + dy^2)^{1/2}$$

$$= 2\pi y \left[1 + \left(\frac{dy}{dx} \right)^2 \right]^{1/2} dx.$$

Then the surface generated between fixed points (x_1, y_1) and (x_2, y_2) has area

$$J = \int_{x_1}^{x_2} 2\pi y \left[1 + \left(\frac{dy}{dx} \right)^2 \right]^{1/2} dx. \quad \dots(1)$$

Comparing equation (1) of § 15.5 (a) with (1), viz. $J = \int f(y, y', x) dx = \text{extremum}$, we get

$$f = 2\pi y \left[1 + \left(\frac{dy}{dx} \right)^2 \right]^{1/2} dx, \quad \dots(2)$$

which gives

$$\frac{\partial f}{\partial x} = 0,$$

so that we may apply equation (2) of Ex. 19 directly

$$\left[\text{i.e. } f - y' \frac{\partial f}{\partial y'} = c \right] \quad \dots(3)$$

Equation (2) gives

$$\frac{\partial f}{\partial y'} = \frac{2\pi y y'}{(1 + y'^2)^{1/2}}, \text{ where } y' = \frac{dy}{dx}.$$

Then equation (3) gives

$$2\pi y [1 + y'^2]^{1/2} - \frac{2\pi y y'^2}{(1 + y'^2)^{1/2}} = c \quad \text{where } c \text{ is constant}$$

or

$$y [1 + y'^2]^{1/2} - \frac{y y'^2}{(1 + y'^2)^{1/2}} = \frac{c}{2\pi} = c_1 \text{ (say)}$$

or

$$\frac{y}{(1 + y'^2)^{1/2}} = c_1$$

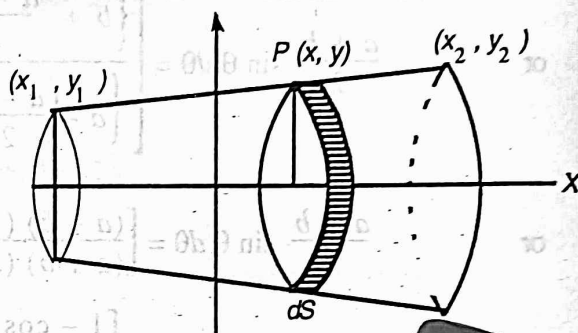


Fig. 15-13

Squaring, we get

$$\frac{y^2}{1+y^2} = c_1^2$$

or

$$y^2 = c_1^2 + c_1^2 y^2$$

or

$$y^2 = \frac{y^2 - c_1^2}{c_1^2}$$

or

$$y' = \frac{\sqrt{(y^2 - c_1^2)}}{c_1}$$

or

$$dy = \frac{\sqrt{(y^2 - c_1^2)}}{c_1} \cdot dx$$

or

$$\frac{dy}{\sqrt{(y^2 - c_1^2)}} = \frac{dx}{c_1}$$

Integrating, we get $y = c_1 \cosh \left(\frac{x}{c_1} + c_2 \right)$... (3)

where c_2 is the constant of integration. The values of c_1 and c_2 are determined by applying the conditions that the curve represented by equation (3) must pass through the points (x_1, y_1) and (x_2, y_2) . Equation (3) represents the required curve known as a *catenary of revolution* of a *catenoid*.

15.7 (c) Hamilton's Principle.

According to Hamilton's principle, "*The path actually traversed in a conservative, holonomic dynamical system from time t_1 and t_2 is one over which the integral of the Lagrangian between limits t_1 and t_2 is stationary (i.e., the time integral of the Lagrangian is extremum.)*"

Analytically it can be represented as

$$\int_{x_1}^{x_2} L \, dt = J = \text{extremum}, \quad \dots (1)$$

where J is the extremum value of the time integral of the Lagrangian and is known as *Hamilton's principle function for the path*.

Equation (1) may be represented as

$$\delta \int_{t_1}^{t_2} L \, dt = 0. \quad \dots (2)$$

where δ is the variation symbol.

This principle helps us to distinguish the actual path from the neighbouring paths.

Deduction of Hamilton's Principle

Let us consider that the conservative holonomic dynamical system moves from P to Q where P and Q are initial and final configurations of the system at times t_1 and t_2 respectively. Let PRQ be the actual path and $PR'Q$, $PR''Q$ the two neighbouring paths out of infinite number of possibilities.

For the deduction of Hamilton's principle the following two conditions must be satisfied:

(i) δt must be equal to zero at end points, i.e., at t_1 the particle must be at P and at t_2 particle must be at Q .

(ii) δr must be equal to zero at end points, i.e., the points P and Q are fixed in space.

Let the system be acted upon by a number of forces represented by F . Let i th particle of the system acted by force F_i acquire acceleration $\ddot{r}_i$, so that we have

$$F_i = m_i \ddot{r}_i.$$

From D' Alembert's principle, we have

$$\sum_i (F_i - m_i \ddot{r}_i) \cdot \delta r_i = 0$$

$$\text{or} \quad \sum_i F_i \cdot \delta r_i - \sum_i m_i \ddot{r}_i \cdot \delta r_i = 0$$

$$\text{But} \quad \ddot{r}_i \cdot \delta r_i = \frac{d}{dt} (\dot{r}_i \cdot \delta r_i) - \dot{r}_i \cdot \frac{d}{dt} (\delta r_i). \quad \dots(2)$$

If there is a little variation along the actual and neighbouring paths, we have

$$\delta r_i = r'_i - r_i \text{ (say).}$$

$$\text{Then} \quad \frac{d}{dt} (\delta r_i) = \frac{d}{dt} (r'_i - r_i) = \frac{dr'_i}{dt} - \frac{dr_i}{dt} = \delta \left(\frac{dr_i}{dt} \right) = \delta (\dot{r}_i). \quad \dots(3)$$

Here primes have been used for neighbouring paths.

Using (3), (2) may be written as

$$\ddot{r}_i \cdot \delta r_i = \frac{d}{dt} (\dot{r}_i \cdot \delta r_i) - \dot{r}_i \cdot \delta (\dot{r}_i)$$

Using above equation, (1) becomes

$$\sum_i F_i \cdot \delta r_i - \sum_i m_i \left[\frac{d}{dt} (\dot{r}_i \cdot \delta r_i) - \dot{r}_i \cdot \delta (\dot{r}_i) \right] = 0$$

$$\text{or} \quad \sum_i F_i \cdot \delta r_i - \sum_i m_i \left[\frac{d}{dt} (\dot{r}_i \cdot \delta r_i) - \frac{1}{2} \delta (\dot{r}_i)^2 \right] = 0$$

$$\text{or} \quad \sum_i F_i \cdot \delta r_i - \sum_i \frac{1}{2} m_i \delta (\dot{r}_i)^2 = \sum_i \frac{d}{dt} (m_i \dot{r}_i \cdot \delta r_i)$$

$$\text{or} \quad \sum_i F_i \cdot \delta r_i + \delta \left(\sum_i \frac{1}{2} m_i \dot{r}_i^2 \right) = \sum_i \frac{d}{dt} (m_i \dot{r}_i \cdot \delta r_i) \quad \dots(5)$$

But $\sum_i F_i \cdot \delta r_i = \text{work done by the forces } F_i \text{ during displacements } \delta r_i = \delta W \text{ (say).}$

and $\sum_i \frac{1}{2} m_i \dot{r}_i^2 = \text{kinetic energy of the system} = T.$

Therefore equation (5) becomes

$$\delta W + \delta T = \sum_i \frac{d}{dt} (m_i \dot{r}_i \cdot \delta r_i).$$

Integrating above expression between the limits t_1 and t_2 , we get

$$\int_{t_1}^{t_2} (\delta W + \delta T) dt = \int_{t_1}^{t_2} \sum_i \frac{d}{dt} (m_i \dot{r}_i \cdot \delta r_i) dt = \sum_i \int_{t_1}^{t_2} d(m_i \dot{r}_i \cdot \delta r_i)$$

$$= \sum_i \left[m_i \dot{r}_i \cdot \delta r_i \right]_P^Q$$

$= 0$ since $\delta r_i = 0$ at the end points P and Q

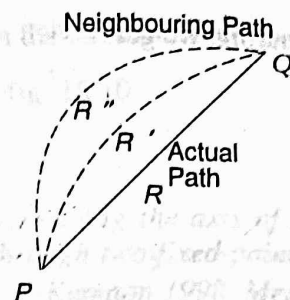


Fig. 15-14

For a conservative system, we know

$\delta W = -\delta V$ where V is the potential energy.

$$\therefore \int_{t_1}^{t_2} (-\delta V + \delta T) dt = 0$$

$$\text{or} \int_{t_1}^{t_2} \delta (T - V) = 0$$

$$\text{i.e.,} \quad \delta \int_{t_1}^{t_2} (T - V) = 0$$

$$\text{or} \quad \delta \int_{t_1}^{t_2} L dt = 0$$

$$\text{or} \quad \int_{t_1}^{t_2} L dt = J = \text{extremum}$$

which is *Hamilton's principle*.

Ex. 26. Use Hamilton's principle to find the equation of motion of one-dimensional harmonic oscillator. (Meerut 1992, Garhwal 1990)

Solution. The kinetic energy of harmonic oscillator,

$$T = \frac{1}{2} m \dot{x}^2$$

The potential energy of harmonic oscillator,

$$V = -\int F dx = \int kx dx = \frac{1}{2} kx^2$$

$\therefore$ The Lagrangian of the system

$$L = T - V = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} kx^2$$

According to the Hamilton's principle

$$\delta \int_{t_1}^{t_2} L dt = 0$$

$$\text{or} \quad \delta \int_{t_1}^{t_2} \left(\frac{1}{2} m \dot{x}^2 - \frac{1}{2} kx^2 \right) dt = 0 \quad \dots(1)$$

$$\text{or} \quad \delta \int_{t_1}^{t_2} \left(\frac{1}{2} m \dot{x}^2 - \frac{1}{2} kx^2 \right) dt = 0$$

$$\delta \int_{t_1}^{t_2} (m \dot{x} \delta \dot{x} - kx \delta x) dt = 0$$

But $\delta \dot{x} = \frac{d}{dt}(\delta x)$ as in equation (3) of section 15-5c

$$\text{or} \quad \int_{t_1}^{t_2} m \dot{x} \frac{d}{dt}(\delta x) dt - \int_{t_1}^{t_2} kx \delta x dt = 0$$

$$\text{or} \quad \left[m \dot{x} \delta x \right]_{t_1}^{t_2} - \int_{t_1}^{t_2} m \frac{d}{dt}(\dot{x}) \delta x dt - \int_{t_1}^{t_2} kx \delta x dt = 0 \quad \dots(2)$$

But $\delta x = 0$ at fixed points, i.e., at instants t_1 and t_2 .

$$\therefore \left[m \dot{x} \delta x \right]_{t_1}^{t_2} = 0.$$

Then equation (2) gives

$$-\int_{t_1}^{t_2} m \frac{d}{dt}(\dot{x}) \delta x dt - \int_{t_1}^{t_2} kx \delta x dt = 0$$

$$\text{i.e.,} \quad \int_{t_1}^{t_2} (m\ddot{x} + kx) \delta x dt = 0.$$

As δx is arbitrary, the above equation is satisfied only if

$$m\ddot{x} + kx = 0,$$

which is the equation of motion for one-dimensional harmonic oscillator.

Ex. 27. A particle of mass m falls a given distance z_0 in time $t_0 = \sqrt{\left(\frac{2z_0}{g}\right)}$ and the distance travelled in time t is given by $z = at + bt^2$, where constants a and b are such that the time t_0 is always the same. Show that the integral $\int_0^{t_0} L dt$ is an extremum for real values of the coefficients only when $a = 0$ and $b \equiv g/2$. (Rohilkhand 2000)

Solution. Let z be distance travelled by the particle after time t ; then

$$\text{kinetic energy of the particle} = \frac{1}{2} m \dot{z}^2,$$

$$\text{potential energy of the particle} = -mgz.$$

The Lagrangian of the system is given by

$$L = T - V = \frac{1}{2} m \dot{z}^2 + mgz.$$

According to Hamilton's principle, we have

$$\delta \int_0^{t_0} L dt = 0$$

or

$$\delta \int_0^{t_0} \left(\frac{1}{2} m \dot{z}^2 + mgz \right) dt = 0$$

or

$$\int_0^{t_0} \left(\frac{1}{2} m \dot{z}^2 + mgz \right) dt = \text{extremum} \quad (A)$$

or

$$\int_0^{t_0} f(z, \dot{z}, t) dt = \text{extremum} \quad (1)$$

where $f(z, \dot{z}, t) = \frac{1}{2} m \dot{z}^2 + mgz$

so that $\frac{\partial f}{\partial z} = +mg, \frac{\partial f}{\partial \dot{z}} = m\dot{z}$

Eq. (1) is correct only if the function f satisfies the Euler-Lagrange equation given by

$$\frac{d}{dt} \left(\frac{\partial f}{\partial \dot{z}} \right) - \frac{\partial f}{\partial z} = 0$$

or $\frac{d}{dt} (m\dot{z}) - mg = 0$

or $m\ddot{z} - mg = 0$

But $z = at + bt^2$... (2)

so that $\dot{z} = a + 2bt$

and $\ddot{z} = 2b$

With this value of $\ddot{z}$, eq. (2) becomes

$$2mb - mg = 0$$

or $b = \frac{g}{2}$... (3)

Also when $z = z_0, t = t_0$, we have

$$z_0 = at_0 + bt_0^2$$

or $\frac{gt_0^2}{2} = at_0 + bt_0^2$ [since $z_0 = \frac{gt_0^2}{2}$ (given)]

$$= at_0 + \frac{gt_0^2}{2} \text{ since } b = \frac{g}{2} \text{ from (3)}$$

which gives $a = 0$

Hence $a = 0$ and $b = \frac{g}{2}$

15.7 (d) Derivation of Lagrange's Equation from Hamilton's Principle.

The Lagrangian function is given by

$$L = L(q_1, q_2, q_3, \dots, q_k, \dots, \dot{q}_1, \dot{q}_2, \dot{q}_3, \dots, \dot{q}_k, \dots, t) \quad (3)$$

In brief we may write $L = L(q, \dot{q}, t)$

If the Lagrangian does not depend upon time explicitly, we have

$$L = L(q, \dot{q})$$

$$\therefore \delta L = \sum_k \frac{\partial L}{\partial q_k} \delta q_k + \sum_k \frac{\partial L}{\partial \dot{q}_k} \delta \dot{q}_k$$

Integrating, we get

$$\int_{t_1}^{t_2} \delta L dt = \int_{t_1}^{t_2} \sum_k \frac{\partial L}{\partial q_k} \delta q_k dt + \int_{t_1}^{t_2} \sum_k \frac{\partial L}{\partial \dot{q}_k} \delta \dot{q}_k dt. \quad \dots(4)$$

(1) But according to Hamilton's principle,

$$\delta \int_{t_1}^{t_2} L dt = 0$$

$$\int_{t_1}^{t_2} \sum_k \frac{\partial L}{\partial q_k} \delta q_k dt + \int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{q}_k} \delta \dot{q}_k dt = 0$$

or

$$\int_{t_1}^{t_2} \sum_k \frac{\partial L}{\partial q_k} \delta q_k dt + \int_{t_1}^{t_2} \sum_k \frac{\partial L}{\partial \dot{q}_k} \frac{d}{dt} (\delta q_k) dt = 0$$

because $\delta \dot{q}_k = \frac{d}{dt} (\delta q_k)$ as in (3) of § 15.5 (c)

or

$$\int_{t_1}^{t_2} \sum_k \frac{\partial L}{\partial q_k} \delta q_k dt + \sum_k \left[\left\{ \frac{\partial L}{\partial \dot{q}_k} \delta q_k \right\}_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) \delta q_k dt \right] = 0 \quad \dots(5)$$

But $\left[\frac{\partial L}{\partial \dot{q}_k} \delta q_k \right]_{t_1}^{t_2} = 0$ because $\delta q_k = 0$ at the end points according to second condition of

Hamilton's principle.

Therefore (5) becomes

$$\int_{t_1}^{t_2} \sum_k \frac{\partial L}{\partial q_k} \delta q_k dt = \sum_k \int_{t_1}^{t_2} \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) \delta q_k dt = 0$$

or

$$\int_{t_1}^{t_2} \sum_k \left[\frac{\partial L}{\partial q_k} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) \right] \delta q_k dt = 0. \quad \dots(6)$$

But variables being independent, the variation δq_k are independent if and only if

$$\frac{\partial L}{\partial q_k} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) = 0,$$

i.e. $\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} = 0, k = 1, 2, 3, \dots$

which is Lagrange's Equation.

15.7 (e) Modified Hamilton's principle.

According to Hamilton's principle, we have

$$\delta J = \delta \int_{t_1}^{t_2} L(q, \dot{q}, t) dt = 0. \quad \dots(1)$$

The relation between Lagrangian and Hamiltonian from equation (5) of § (15.4 b) is given by

$$H(q, p, t) = \sum_k p_k \dot{q}_k - L(q, \dot{q}, t).$$

$$\therefore L(q, \dot{q}, t) = \sum_k p_k \dot{q}_k - H(q, p, t).$$

Then equation (1) becomes

$$\delta J = \delta \int_{t_1}^{t_2} [\sum_k p_k \dot{q}_k - H(q, p, t)] dt = 0. \quad \dots(2)$$

This is called modified Hamilton's principle.

15.7 (f) Derivation of Hamilton's Equations from Modified Hamilton's Principle (i.e., Variational Principle)

Consider two paths PRQ and $PR'Q$ out of infinite number of possibilities as shown in fig. 15-15. In this case δ variation comprises independent variations of both q_k and p_k at constant time t .

The difference between the two paths for the given value of t , may be described by introducing a parameter α common to all points of the path of integration in phase space.

If $\bar{q}_k, \bar{p}_k$ are the values of q_k and p_k , for the varied path $PR'Q$, we have

$$\bar{q}_k = q_k + \delta q_k = q_k + \frac{\partial q_k}{\partial \alpha} \delta \alpha = q_k + \eta_k \delta \alpha \quad \dots(1)$$

$$\text{and} \quad \bar{p}_k = p_k + \delta p_k = p_k + \frac{\partial p_k}{\partial \alpha} \delta \alpha = p_k + \xi_k \delta \alpha, \quad \dots(2)$$

$$\text{where } \frac{\partial q_k}{\partial \alpha} = \eta_k \text{ and } \frac{\partial p_k}{\partial \alpha} = \xi_k. \quad \dots(3)$$

The variations in q_k and p_k are given by

$$\left. \begin{aligned} \delta q_k &= \eta_k \delta \alpha, \\ \delta p_k &= \xi_k \delta \alpha, \end{aligned} \right\} \quad \dots(4)$$

from (1) and (2)

If $\delta \alpha = 0$, then $\delta q_k = \delta p_k = 0$

so that $\bar{q}_k = q_k$ and $\bar{p}_k = p_k$, i.e., the varied path $PR'Q$ coincides with the actual path PRQ .

As the times at end points are not varied so that

$$\left. \begin{aligned} \eta_k(t_1) &= \eta_k(t_2) = 0 \\ \xi_k(t_1) &= \xi_k(t_2) = 0. \end{aligned} \right\} \quad \dots(5)$$

In terms of parameter α , Hamilton's principle becomes

$$\delta J = \frac{\partial J}{\partial \alpha} \delta \alpha = \delta \alpha \frac{\partial}{\partial \alpha} \int_{t_1}^{t_2} \sum_k p_k \dot{q}_k - H(q, p, t) dt = 0 \quad \dots(6)$$

As the times at the end points are not varied and hence are not the functions of α so that the differentiation and integration may be interchanged. Then equation (6) may be written as

$$\delta \alpha \int_{t_1}^{t_2} \frac{\partial}{\partial \alpha} [\sum_k p_k \dot{q}_k - H(q, p, t)] dt = 0$$

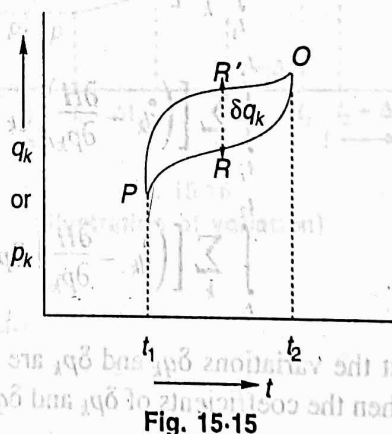


Fig. 15-15

or
$$\delta\alpha \int_{t_1}^{t_2} \sum_k \left[\frac{\partial p_k}{\partial \alpha} \dot{q}_k + \frac{\partial \dot{q}_k}{\partial \alpha} p_k - \frac{\partial H}{\partial q_k} \frac{\partial q_k}{\partial \alpha} - \frac{\partial H}{\partial p_k} \frac{\partial p_k}{\partial \alpha} \right] dt = 0. \quad \dots(7)$$

But
$$\begin{aligned} \int_{t_1}^{t_2} \frac{\partial \dot{q}_k}{\partial \alpha} p_k dt &= \int_{t_1}^{t_2} p_k \frac{d}{dt} \left(\frac{\partial q_k}{\partial \alpha} \right) dt \\ &= \left[p_k \frac{\partial q_k}{\partial \alpha} \right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \dot{p}_k \frac{\partial q_k}{\partial \alpha} dt \text{ integrating by parts} \\ &= \left[p_k \eta_k \right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \eta_k \dot{p}_k dt \quad [\text{from (3)}] \\ &= - \int_{t_1}^{t_2} \eta_k \dot{p}_k dt \quad \dots(8) \end{aligned}$$

since $\left[p_k \eta_k \right]_{t_1}^{t_2} = p_k \eta_k(t_2) - p_k \eta_k(t_1) = 0$, [from (5)]

Then using (3) and (8), equation (7) becomes

$$\delta\alpha \int_{t_1}^{t_2} \sum_k \left[\xi_k \dot{q}_k - \eta_k \dot{p}_k - \frac{\partial H}{\partial q_k} \eta_k - \frac{\partial H}{\partial p_k} \xi_k \right] dt = 0$$

or
$$\int_{t_1}^{t_2} \sum_k \left[\dot{q}_k \xi_k \delta\alpha - \eta_k \dot{p}_k \delta\alpha - \frac{\partial H}{\partial q_k} \eta_k \delta\alpha - \frac{\partial H}{\partial p_k} \xi_k \delta\alpha \right] dt = 0$$

or
$$\int_{t_1}^{t_2} \sum_k \left[\left(\dot{q}_k - \frac{\partial H}{\partial p_k} \right) \xi_k \delta\alpha - \left(\dot{p}_k + \frac{\partial H}{\partial q_k} \right) \eta_k \delta\alpha \right] dt = 0$$

or
$$\int_{t_1}^{t_2} \sum_k \left[\left(\dot{q}_k - \frac{\partial H}{\partial p_k} \right) \delta p_k - \left(\dot{p}_k + \frac{\partial H}{\partial q_k} \right) \delta q_k \right] dt = 0 \quad \dots(9)$$

[from (4)]

But the variations δq_k and δp_k are independent of each other ; the integral (9) is satisfied only when the coefficients of δp_k and δq_k vanish separately, i.e., when

$$\dot{q}_k - \frac{\partial H}{\partial p_k} = 0 \text{ and } \dot{p}_k + \frac{\partial H}{\partial q_k} = 0$$

i.e.,
$$\frac{\partial H}{\partial p_k} = \dot{q}_k \text{ and } \frac{\partial H}{\partial q_k} = -\dot{p}_k \quad \dots(10)$$

which are required Hamilton's equations.

15.7 (g) Principle of Least Action

The time integral of twice the kinetic energy is called the *action*.

The principle of least action states that the variation of action along the actual path between given time interval is least i.e.,

$$\Delta \int_{t_1}^{t_2} 2T dt = 0. \quad \dots(1)$$

But in system for which Hamiltonian H remains constant,

$$2T = \sum_k p_k \dot{q}_k$$

Therefore for such systems, the principle of least action may be written as

$$\Delta \int_{t_1}^{t_2} \sum_k p_k \dot{q}_k dt = 0 \quad \dots(2)$$

where Δ represents a new type of variation of the path which allows time as well as position co-ordinates to vary.

Deduction of principle of least action

For the deduction of principle of least action we have to use a variation termed as Δ variation in which

- (i) Time as well as the position co-ordinates are allowed to vary.
- (ii) Time t varies even at the end points of the path.
- (iii) The position co-ordinates are held fixed at the end points of the path i.e., $\Delta q_k = 0$ at the end points.

Let PRQ be the actual path and $P'R'Q'$, the varied path. The end points P and Q after time Δt take the positions P' and Q' such that the position co-ordinates of P and Q are fixed while the time t is not fixed. A point R on the actual path now goes over R' on the varied path with the correspondence.

$$q_k \rightarrow q_k' = q_k + \Delta q_k.$$

If α is variational parameter, then in δ process t is independent of α ; but in Δ process t is function of α even at the end points, i.e., $t = t(\alpha)$. Thus the function q_k depends on t and α throughout.

Analytically Δ variation is defined as

$$\begin{aligned} \Delta q_k &= \left[\frac{d}{d\alpha} q_k(\alpha, t) \right] = \left[\frac{\partial q_k}{\partial \alpha} + \frac{\partial q_k}{\partial t} \frac{dt}{d\alpha} \right] d\alpha \\ &= \frac{\partial q_k}{\partial \alpha} d\alpha + \dot{q}_k \frac{dt}{d\alpha} d\alpha. \end{aligned}$$

$$\begin{aligned} \text{But } \delta q_k &= \frac{\partial q_k}{\partial \alpha} d\alpha \text{ as in (1) of § 15.5 (f)} \\ \text{and } \dot{q}_k \frac{dt}{d\alpha} d\alpha &= \dot{q}_k \Delta t \end{aligned} \quad \dots(1)$$

$$\text{Hence } \Delta q_k = \delta q_k + \dot{q}_k \Delta t \quad \dots(2)$$

Analytically Δ -variation of any $f(q_k, \dot{q}_k, t)$ is given by

$$\Delta f = \sum_k \left(\frac{\partial f}{\partial q_k} \Delta q_k + \frac{\partial f}{\partial \dot{q}_k} \Delta \dot{q}_k + \frac{\partial f}{\partial t} \Delta t \right)$$

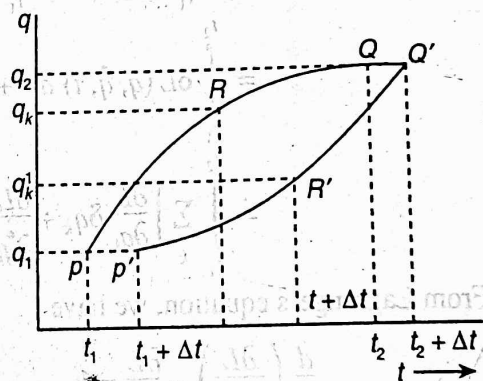


Fig. 15-16
(Illustration of variation)

$$\begin{aligned}
 &= \sum_k \left(\frac{\partial f}{\partial q_k} (\delta q_k + \dot{q}_k \Delta t) + \sum_k \frac{\partial f}{\partial \dot{q}_k} (\delta \dot{q}_k + \ddot{q}_k \Delta t) + \frac{\partial f}{\partial t} \Delta t \right) \\
 &= \delta f + \frac{df}{dt} \Delta t
 \end{aligned}$$

Thus $\Delta \equiv \delta + \Delta t \frac{d}{dt}$.

It is to be noted that the Δ operation and time differentiation can not be interchanged in this case which is allowed in the case of δ variation.

Hamilton's principle is

$$J = \int_{t_1}^{t_2} L dt = \text{extremum.}$$

Taking Δ variation, we have

$$\begin{aligned}
 \Delta J &= \Delta \int_{t_1}^{t_2} L dt = \left(\delta + \Delta t \frac{d}{dt} \right) \int_{t_1}^{t_2} L dt \text{ from (3)} \\
 &= \delta \int_{t_1}^{t_2} L dt + \Delta t \int_{t_1}^{t_2} d(L) = \delta \int_{t_1}^{t_2} L dt + \left[L \Delta t \right]_{t_1}^{t_2} \\
 &= \int_{t_1}^{t_2} \delta L(q, \dot{q}, t) dt + \left[L \Delta t \right]_{t_1}^{t_2} \\
 &= \int_{t_1}^{t_2} \sum_k \left[\frac{\partial L}{\partial q_k} \delta q_k + \frac{\partial L}{\partial \dot{q}_k} \delta \dot{q}_k \right] dt + \left\{ L \Delta t \right\}_{t_1}^{t_2} \dots (4)
 \end{aligned}$$

From Lagrange's equation, we have

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} = 0$$

or

$$\frac{\partial L}{\partial q_k} = \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) \dots (5)$$

Also we know $\delta \dot{q}_k = \frac{d}{dt} (\delta q_k)$... (6)

Using (5) and (6), equation (4) becomes

$$\begin{aligned}
 \Delta \int_{t_1}^{t_2} L dt &= \int_{t_1}^{t_2} \sum_k \left[\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) \delta q_k + \frac{\partial L}{\partial \dot{q}_k} \frac{d}{dt} (\delta q_k) \right] dt + \left[L \Delta t \right]_{t_1}^{t_2} \\
 &= \int_{t_1}^{t_2} \sum_k \left[\dot{p}_k \delta q_k + p_k \frac{d}{dt} (\delta q_k) \right] dt + \left[L \Delta t \right]_{t_1}^{t_2} \left(\text{since } p_k = \frac{\partial L}{\partial \dot{q}_k} \right) \\
 &= \int_{t_1}^{t_2} \sum_k \frac{d}{dt} (p_k \delta q_k) dt + \left[L \Delta t \right]_{t_1}^{t_2}
 \end{aligned}$$

$$\begin{aligned}
 &= \int_{t_1}^{t_2} \sum_k \frac{d}{dt} \{p_k (\Delta q_k - \dot{q}_k \Delta t)\} dt + \left[L \Delta t \right]_{t_1}^{t_2} \text{ using (2)} \\
 &= \int_{t_1}^{t_2} \sum d(p_k \Delta q_k) - \int_{t_1}^{t_2} \sum d(p_k \dot{q}_k \Delta t) + \left[L \Delta t \right]_{t_1}^{t_2} \\
 &= \sum_k \left[p_k \Delta q_k \right]_{t_1}^{t_2} - \sum_k \left[p_k \dot{q}_k \Delta t \right]_{t_1}^{t_2} + \left[L \Delta t \right]_{t_1}^{t_2} \dots (7)
 \end{aligned}$$

But $\left[p_k \Delta q_k \right]_{t_1}^{t_2} = 0$ since $\Delta q_k = 0$ at end points.

$\therefore$ Equation (7) becomes

$$\Delta \int_{t_1}^{t_2} L dt = \left[(L - \sum p_k \dot{q}_k \Delta t) \right]_{t_1}^{t_2} = - \left[H \Delta t \right]_{t_1}^{t_2} \quad \left(\text{since } H = \sum p_k \dot{q}_k - L \right) \dots (8)$$

If we restrict ourselves to systems for which $\frac{\partial H}{\partial t} = 0$, i.e., for which H remains constant, then

$$= \left[H \Delta t \right]_{t_1}^{t_2} = \Delta \int_{t_1}^{t_2} H dt$$

Substituting this in (8), we get

$$\begin{aligned}
 \Delta \int_{t_1}^{t_2} L dt &= - \Delta \int_{t_1}^{t_2} H dt \\
 \text{or} \quad \Delta \int_{t_1}^{t_2} (L + H) dt &= 0 \dots (9)
 \end{aligned}$$

$$\begin{aligned}
 \text{or} \quad \Delta \int_{t_1}^{t_2} (L + \sum_b p_b \dot{q}_b - L) dt &= 0 \\
 \text{or} \quad \Delta \int_{t_1}^{t_2} \sum p_b \dot{q}_b dt &= 0 \dots (10)
 \end{aligned}$$

This is principle of least action.

The quantity $\int_{t_1}^{t_2} \sum p_k \dot{q}_k$ is generally called *Hamilton's characteristic function*.

$$\text{But} \quad \sum_k p_k \dot{q}_k = 2T.$$

$$\therefore \Delta \int_{t_1}^{t_2} 2T dt = 0 \dots (11)$$

which is another form of principle of least action.

Note. For deducing the principle of least action, Δ variation applied is represented by equation (3) with restricting ourselves to the systems for which Hamiltonian H is constant of motion and takes the same value on the varied path as on the actual path.

The principle of least action may also be stated as follows :

The path of the system in configuration space is that along which the integral energy is extremum or stationary when compared with that along neighbouring paths satisfying the given conditions provided the initial and final configuration of the system are given and the Hamiltonian is independent of the time.

15.7 (h) Principle of least action in terms of arc length of the particle trajectory

Let a system contain only one particle of mass m . The kinetic energy of the particle is given by

$$T = \frac{1}{2} m \left(\frac{ds}{dt} \right)^2 \quad \dots(1)$$

where ds is the element of arc traversed by the particle in time dt . Equation (1) yields

$$dt = \sqrt{\left(\frac{m}{2T} \right)} ds$$

so that the principle of least action, from eqn. (1) of § 12.5 (f) may be expressed as

$$\Delta \int_{t_1}^{t_2} 2T dt = \Delta \int \sqrt{\left(\frac{m}{2T} \right)} ds = 0 \quad \text{from (2)}$$

$$\text{or} \quad \Delta \int \sqrt{2mT} ds = 0$$

$$\text{or} \quad \Delta \int \sqrt{[2m(E - V)]} ds = 0 \quad E \text{ being the total energy of the particle}$$

$$\text{or} \quad \Delta \int \sqrt{[2m(H - V)]} ds = 0, \quad \text{since } E = H$$

$$\text{or} \quad \Delta \int \sqrt{(H - V)} ds = 0, \quad \dots(3)$$

(since $m = \text{constant}$)

Equation (3) represents the principle of least action in terms of arc length of the particle trajectory.

15.7 (i) Jacobi's form of the principle of least action

The kinetic energy of the system in generalised co-ordinates is expressed as

$$T = \frac{1}{2} \sum_{i,k} m_{ik} \dot{q}_i \dot{q}_k = \frac{1}{2} \sum_{i,k} m_{ik} \frac{dq_i dq_k}{dt^2} \quad \dots(1)$$

Let us define a new quantity p given by the equation

$$(dp)^2 = \sum m_{ik} dq_i dq_k \quad \dots(2)$$

Then equation (1) becomes

$$T = \frac{1}{2} \left(\frac{dp}{dt} \right)^2$$

which yields

$$dt = \frac{dp}{\sqrt{2T}}$$

so that the principle of least action, eqn. (1) of § 15.5 g, may be expressed as

$$\Delta \int_{t_1}^{t_2} 2T dt = \Delta \int_{t_1}^{t_2} 2T \frac{dp}{\sqrt{(2T)}} = 0$$

$$\Delta \int_{t_1}^{t_2} \sqrt{(2T)} dp = 0$$

or

$$\Delta \int_{t_1}^{t_2} \sqrt{[2(E - V)]} dp = 0$$

or

$$\Delta \int_{t_1}^{t_2} \sqrt{[(H - V)]} dp = 0.$$

This equation represents the *Jacobi's form of the principle of least action*.

Ex. 28. Apply principle that the time integral of least action to prove that out of all possible paths between two points, the system for which kinetic energy is conserved moves along the path for which the transit time is extremum. (Kanpur 2006)

Solution. According to principle of least action, we have

$$\Delta \int_{t_1}^{t_2} 2T dt = 0.$$

If kinetic energy (T) of the system is conserved, then above equation yields

$$\Delta \int_{t_1}^{t_2} dt = 0$$

or

$$\Delta (t_2 - t_1) = 0,$$

which states that the system moves along the path for which the transit time is extremum.

Ex. 29. State Fermat's principle in geometrical optics and apply the principle of least action to prove it.

Solution. Fermat's Principle. It states that the time taken by a light ray to travel between two points is extremum.

Proof. According to principle of least action,

$$\Delta \int_{t_1}^{t_2} 2T dt = 0,$$

which yields for constant T ,

$$\Delta (t_2 - t_1) = 0$$

or

$$(t_2 - t_1) = \text{extremum},$$

i.e., the time taken by the light ray to travel between two points is extremum.

15.7 (j) Deduction of Lagrange's equation for non-conservative systems from the extended Hamilton's principle.

We can generalize Hamilton's principle so that it may include non-conservative forces and may be used to derive Lagrange's equation for non-conservative systems.

Hamilton's principle with fixed end points is

$$\delta J = \delta \int_{t_1}^{t_2} L dt = 0.$$

If it is put in the form

$$\delta J = \delta \int_{t_1}^{t_2} (T + W) dt = 0 \quad \dots(1)$$

where

$$\delta W = \sum_i \mathbf{F}_i \cdot \delta \mathbf{r}_i, \quad \dots(2)$$

where δW represents the work done by the forces on the system during the virtual displacement from the actual to the varied path.

Equation (1) represents that the time integral of the variation of the kinetic energy plus the virtual work done during the virtual displacement from actual to varied path must be zero, which is extended Hamilton's principle.

We have
$$\sum_i \mathbf{F}_i \cdot \delta \mathbf{r}_i = \sum_i \sum_k \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} \delta q_k, \quad \dots(3)$$

when q_k are generalised co-ordinates, since the variation, $\delta \mathbf{r}_i$ or δq_i are identical with virtual displacements of co-ordinates as there is no variation of time involved

Substituting
$$\sum_i \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} = Q_k, \quad \dots(4)$$

where Q_k is termed as generalised force.

Using (4), (3) becomes

$$\sum_i \mathbf{F}_i \cdot \delta \mathbf{r}_i = \sum_k Q_k \delta q_k. \quad \dots(5)$$

Using (3) and (5), equation (2) can be written as

$$\delta \int_{t_1}^{t_2} T dt + \int_{t_1}^{t_2} \sum_k Q_k \delta q_k dt = 0. \quad \dots(6)$$

For conservative systems, kinetic energy T like L is function of q_k and $\dot{q}_k$ so that we have

$$\delta \int_{t_1}^{t_2} T dt = \int_{t_1}^{t_2} \sum_k \left[\frac{\partial T}{\partial q_k} - \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_k} \right) \right] \delta q_k dt. \quad \dots(7)$$

Using equation (7), equation (6) may be written as

$$\int_{t_1}^{t_2} \sum_k \left[\frac{\partial T}{\partial q_k} - \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_k} \right) \right] \delta q_k dt + \int_{t_1}^{t_2} \sum_k Q_k \delta q_k dt = 0$$

or
$$\int_{t_1}^{t_2} \sum_k \left[\frac{\partial T}{\partial q_k} - \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_k} \right) + Q_k \right] \delta q_k dt = 0.$$

Since the constraints are holonomic, the displacements δq_k are independent and hence the integral can vanish if and only if

$$\frac{\partial T}{\partial q_k} - \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_k} \right) + Q_k = 0$$

or

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_k} \right) - \frac{\partial T}{\partial q_k} = Q_k, \quad \dots(8)$$

which is Lagrange's equation for holonomic and non-conservative systems.

15.7 (k) Deduction of Lagrange's equation for non-holonomic systems (Method of Lagrangian undetermined multipliers).

In the derivation of Lagrange's equation the holonomic constraints do not appear until δq_k are considered to be independent of each other.

In the non-holonomic systems the generalised co-ordinates q_k are not independent of each other and they cannot be reduced further by means of equations of constraints of the form

$$f(q_1, q_2, \dots, q_n, t) = 0,$$

Let us consider non-holonomic system in which the constraints are represented by m equations of the form

$$\sum_{k=1}^n A_{jk} dq_k + B_j dt = 0, \quad \dots(1)$$

where $j = 1, 2, 3, \dots, m$ and n is the number of generalised co-ordinates so that the number of degrees of freedom $= (n - m)$. According to Hamilton's principle, we have

$$\delta \int_{t_1}^{t_2} L dt = 0$$

which from equation (6) of § 15.7 *d* gives

$$\int_{t_1}^{t_2} \sum_k \left[\frac{\partial L}{\partial q_k} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) \right] \delta q_k dt = 0 \quad \dots(2)$$

It is to be noted that so far the constraints have not been accounted.

In Hamilton's principle the variation process involved is such that the time on each point of the path is held constant. Hence the restriction on the variations due to constraints, from (1), are given by m equations of the form

$$\sum_k A_{jk} dq_k = 0 \quad \dots(3)$$

(since $dt = 0$)

These equations can be used to reduce the number of virtual displacements to independent ones. The method for eliminating extra virtual displacements is called the method of Lagrangian undetermined multipliers

If equation (3) holds then we must have

$$\lambda_j \sum_k A_{jk} dq_k = 0 \quad \dots(4)$$

where λ_j ($j = 1, 2, 3, \dots, m$) are undetermined constants and are permitted to be functions of t , but not the other variables. Summing equations (4) and integrating the resulting equation from t_1 to t_2 , we get

$$\int_{t_1}^{t_2} \sum_{j,k} \lambda_j A_{jk} dq_k dt = 0. \quad \dots(5)$$

Combining (2) and (5), we get

$$\int_{t_1}^{t_2} \sum_k \left\{ \frac{\partial L}{\partial q_k} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) + \sum_j \lambda_j A_{jk} \right\} \delta q_k dt = 0. \quad \dots(6)$$

Here the displacements δq_k 's are still not independent because they are connected by equation (3).

As there are $(n - m)$ degrees of freedom, the first $(n - m)$ of the co-ordinates q_k 's may be considered to be independent while the last m are then fixed by equation (3).

So far the values λ_j 's are unspecified. Let us choose them such that

$$\frac{\partial L}{\partial q_k} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) + \sum_j \lambda_j A_{jk} = 0, \quad \dots(7)$$

for $k = n - m + 1, n - m + 2, \dots, n$.

With the value of λ_j given by (7), equation (6) may be written as

$$\int_{t_1}^{t_2} \sum_k \left\{ \frac{\partial L}{\partial q_k} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) + \sum_j \lambda_j A_{jk} \right\} \delta q_k dt = 0 \quad \dots(8)$$

for $k = 1, 2, 3, \dots, n - m$.

In this equation q_k 's are independent, so that this is only satisfied if the coefficients of δq_k separately vanish, i.e.,

$$\frac{\partial L}{\partial q_k} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) + \sum_j \lambda_j A_{jk} = 0 \quad \text{for } k = 1, 2, 3, \dots, n - m. \quad \dots(9)$$

From (7) and (9), we have

$$\frac{\partial L}{\partial q_k} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) + \sum_j \lambda_j A_{jk} = 0 \quad \text{for } k = 1, 2, 3, \dots, n$$

$$\text{or } \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} = \sum_j \lambda_j A_{jk}, \quad \text{for } k = 1, 2, 3, \dots, n \quad \dots(10)$$

which represents complete set of Lagrange's equation for non-holonomic and conservative system.

Note. If the motion under the applied generalised force components Q_k coincides with that under constraints, then equation (10) must be identical with (9) of 15.5J. i.e.

$$Q_k = \sum_{j=1}^m \lambda_j A_{jk} \quad \dots(11)$$

Ex. 30. Find the equation of motion of a hoop rolling without slipping on an inclined plane and hence find its acceleration and frictional force of constraint.

(Rohilkhand, 2008, 2001, 1986, Agra 1997)

Solution. Let ϕ be the inclination of the inclined plane of length l with the horizontal. Let the hoop of radius r roll down from point O without slipping; then the equation of the constraint is

$$dx = r d\theta$$

or $r d\theta - dx = 0$... (1)

As there is only one equation of the constraint, only one Lagrangian multiplier is required. The coefficients in the constraint equation are

$$A_\theta = r \text{ and } A_x = -1. \quad \dots (2)$$

Kinetic energy of the hoop, T = kinetic energy of the motion of centre of mass + kinetic energy of motion about the centre of mass.

$$\frac{1}{2} M \dot{x}^2 + \frac{1}{2} M r^2 \dot{\theta}^2,$$

The potential energy of the hoop,

$$V = Mgh = Mg(l-x) \sin \phi.$$

so that the Lagrangian of the system given by

$$L = T - V = \frac{1}{2} M (\dot{x}^2 + r^2 \dot{\theta}^2) - Mg(l-x) \sin \phi. \quad \dots (3)$$

As there are only two generalised co-ordinates x and θ , there will be two Lagrangian equations

From (3), we have

$$\left. \begin{aligned} \frac{\partial L}{\partial \dot{x}} &= M \dot{x}, \quad \frac{\partial L}{\partial x} = Mg \sin \phi \\ \frac{\partial L}{\partial \dot{\theta}} &= Mr^2 \dot{\theta}, \quad \frac{\partial L}{\partial \theta} = 0. \end{aligned} \right\} \quad \dots (4)$$

The Lagrangian equation in x is given by

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = A_x \lambda$$

or $M \ddot{x} - Mg \sin \phi + \lambda = 0$ [from (2) and (4)] ... (5)

The Lagrangian equation in θ is given by

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = A_\theta \lambda,$$

i.e. $Mr^2 \ddot{\theta} - r\lambda = 0$. [from (2) and (4).] ... (6)

Equations (5) and (6) represent Lagrangian equations.

Equation (1) gives $r \ddot{\theta} = \ddot{x}$ (7)

Using this, (2) gives $M \ddot{x} = \lambda$ (8)

Substituting this value of λ in (5) we get

$$M \ddot{x} - Mg \sin \phi + M \ddot{x} = 0$$

or $\ddot{x} = \frac{g \sin \phi}{2}$... (9)

Then (7) gives $\ddot{\theta} = \frac{g \sin \phi}{2r}$... (10)

From equation (9) it is clear that the hoop rolls down the inclined plane with only one half the acceleration it would have in slipping down a frictionless inclined plane.

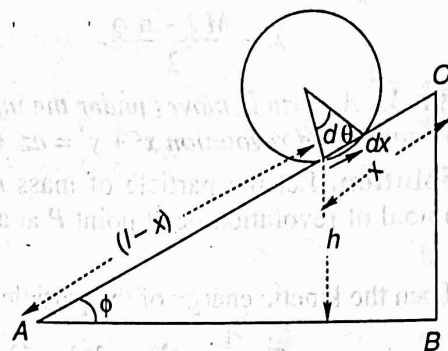


Fig. 15.17

(1). The frictional force of constraint λ , from equations (8) and (9), is given by

$$\lambda = \frac{Mg \sin \phi}{2}.$$

Ex. 31. A particle moves under the influence of gravity on the frictionless inner surface of the paraboloid of revolution $x^2 + y^2 = az$. Obtain the equations of motion.

Solution. Let the particle of mass m moving on the frictionless inner surface of the paraboloid of revolution be at point P at any instant t . Let the cylindrical co-ordinates of P be (r, θ, z) .

Then the kinetic energy of the particle,

$$T = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2 + \dot{z}^2)$$

The potential energy, $V = mgz$ so that the Lagrangian of the system is given by

$$L = T - V = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2 + \dot{z}^2) - mgz \quad \dots(1)$$

The equation of the paraboloid of revolution is

$$x^2 + y^2 = az$$

But $x^2 + y^2 = r^2$; therefore the paraboloid of revolution is expressed as

$$r^2 = az$$

or

$$r^2 - az = 0$$

which gives the condition of constraint

$$2r dr - a dz = 0. \quad \dots(2)$$

$\therefore$ The coefficients in the constraint eqn. (2) are given by

$$A_r = 2r; A_\theta = 0 \text{ and } A_z = -a. \quad \dots(3)$$

Equation (1) gives

$$\left. \begin{aligned} \frac{\partial L}{\partial r} &= m\dot{r}, \quad \frac{\partial L}{\partial r} = mr\dot{\theta}^2 \\ \frac{\partial L}{\partial \dot{r}} &= mr^2\dot{\theta}, \quad \frac{\partial L}{\partial \theta} = 0 \\ \frac{\partial L}{\partial \dot{z}} &= m\dot{z}, \quad \frac{\partial L}{\partial z} = -mg \end{aligned} \right\} \quad \dots(4)$$

The Lagrange's equations are given by

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{r}} \right) - \frac{\partial L}{\partial r} = A_r \lambda,$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = A_\theta \lambda$$

and

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{z}} \right) - \frac{\partial L}{\partial z} = A_z \lambda$$

$$\text{or } m\ddot{r} - mr\dot{\theta}^2 = 2r\lambda$$

$$\left. \begin{aligned} \frac{d}{dt} (mr^2\dot{\theta}) &= 0, \text{ i.e., } m\ddot{r}r + 2mr\dot{r}\dot{\theta} = 0 \\ \text{and } m\ddot{z} + mg &= -a\lambda. \end{aligned} \right\} \quad \text{from (3) and (4)}$$

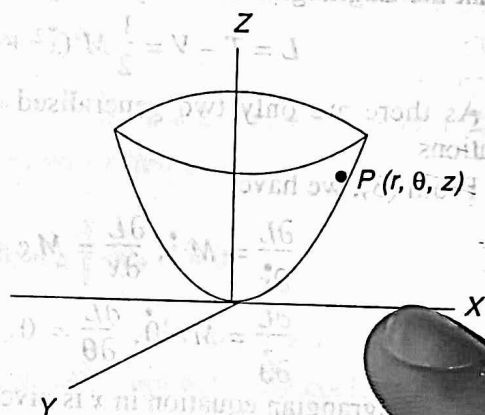


Fig. 15-18

15.8 The Two-Body Central Force Problems

15.8. (a) Reduction of Two-Body Problem to one-body Problem.

A two-body system can be effectively reduced to one-body system by introducing the concept of *reduced mass*. Suppose a system is composed of two masses m_1 and m_2 , then for an inertial observer the relative motion of these masses may be expressed by a *fictitious particle* of *reduced mass* μ .

Let the instantaneous position of masses m_1 and m_2 be represented by position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ relative to an arbitrary origin O (Fig. 15-19).

Let us assume that no external force acts on the system and the internal force is simply due to mutual interaction of the particles, so the potential energy function is dependent only on internal forces, so the potential energy function may be assumed to be the function of vector between two particles $\mathbf{r}_2 - \mathbf{r}_1$ or their relative velocity $(\dot{\mathbf{r}}_2 - \dot{\mathbf{r}}_1)$. This system has six degrees of freedom and hence requires six independent generalised coordinates. We choose these to be three components of radius vector $\mathbf{R}$ of centre of mass and three components of difference vector $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$.

The Lagrangian of the system has the form

$$L = T(\dot{\mathbf{R}}, \dot{\mathbf{r}}) - V(\mathbf{r}, \dot{\mathbf{r}}, \dots) \quad \dots(1)$$

The kinetic energy of the system may be expressed as the sum of kinetic energy of the motion of centre of mass plus the kinetic energy of the motion about the centre of mass, i.e.,

$$T = \frac{1}{2}(m_1 + m_2)\dot{\mathbf{R}}^2 + \left(\frac{1}{2}m_1\dot{\mathbf{r}}_{C_1}^2 + \frac{1}{2}m_2\dot{\mathbf{r}}_{C_2}^2\right) \quad \dots(2)$$

where $\mathbf{r}_{C_1}$ and $\mathbf{r}_{C_2}$ are the radii vectors of the two particles relative to centre of mass. $\mathbf{r}_{C_1}$ and $\mathbf{r}_{C_2}$ are related to $\mathbf{r}$ by the equations

$$\left. \begin{aligned} \mathbf{r}_{C_1} &= -\frac{m_2}{m_1 + m_2} \mathbf{r} \\ \mathbf{r}_{C_2} &= \frac{m_1}{m_1 + m_2} \mathbf{r} \end{aligned} \right\} \quad \dots(3)$$

Substituting these values in (2), we get

$$T = \frac{1}{2}(m_1 + m_2)\dot{\mathbf{R}}^2 + \frac{1}{2}\left(\frac{m_1 m_2}{m_1 + m_2}\right)\dot{\mathbf{r}}^2 \quad \dots(4)$$

so the Lagrangian of the system

$$L = T - V = \frac{1}{2}(m_1 + m_2)\dot{\mathbf{R}}^2 + \frac{1}{2}\left(\frac{m_1 m_2}{m_1 + m_2}\right)\dot{\mathbf{r}}^2 - V(\mathbf{r}, \dot{\mathbf{r}})$$

$\mathbf{R}$ does not occur in Lagrangian, so three components of $\mathbf{R}$ are cyclic, so centre of mass is either at rest or in uniform motion. No equation of motion for $\mathbf{r}$, will contain terms containing $\mathbf{R}$ or $\dot{\mathbf{R}}$, so assuming centre of mass at rest, we may drop the first term from the Lagrangian, i.e.,

$$L = \frac{1}{2}\left(\frac{m_1 m_2}{m_1 + m_2}\right)\dot{\mathbf{r}}^2 - V(\mathbf{r}, \dot{\mathbf{r}}, \dots)$$

This is the Lagrangian of a single particle of a fictitious mass $\mu = \frac{m_1 m_2}{m_1 + m_2}$ placed at the

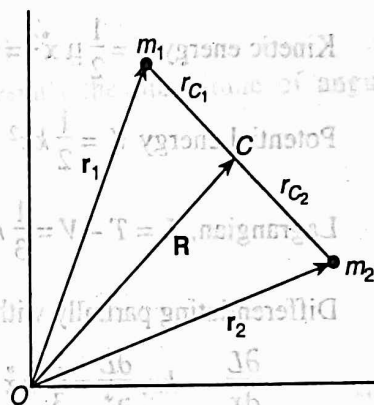


Fig. 5-19

location of m_2 with the original potential function. Thus two body problem is equivalent to one body problem.

Thus the two-body problem is reduced to one-body problem if we consider the motion represented by the motion of a fictitious particle of mass μ placed (at location of m_2) relative to origin at the location of mass m_1 under the original potential energy function V .

Ex. 32. Two rigid bodies of masses ' m ' and ' $2m$ ' are connected by a light flexible spring of spring constant K . Write down the Lagrangian of the system and obtain Lagrange equation of motion. (Bombay 2004, Meerut 2001)

Solution. Let x be compression of spring of any instant. For simplicity we shall take generalised coordinate x and use the concept of reduced mass μ placed at the location of lighter mass ' m ' with same initial force.

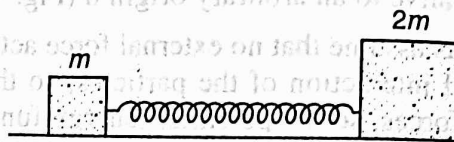


Fig. 15-20

$$\text{Reduced mass } \mu = \frac{m_1 m_2}{m_1 + m_2} = \frac{m(2m)}{m + 2m} = \frac{2}{3} m$$

$$\text{Kinetic energy } T = \frac{1}{2} \mu \dot{x}^2 = \frac{1}{2} \left(\frac{2}{3} m \right) \dot{x}^2 = \frac{1}{3} m \dot{x}^2$$

$$\text{Potential energy } V = \frac{1}{2} kx^2$$

$$\text{Lagrangian, } L = T - V = \frac{1}{3} m \dot{x}^2 - \frac{1}{2} kx^2$$

Differentiating partially with respect to x and $\dot{x}$, we have

$$\frac{\partial L}{\partial x} = -kx, \quad \frac{\partial L}{\partial \dot{x}} = \frac{2}{3} m \dot{x}$$

Lagrange's equation is

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = 0$$

$$\Rightarrow \frac{d}{dt} \left(\frac{2}{3} m \dot{x} \right) + kx = 0$$

$$\Rightarrow \frac{2}{3} m \ddot{x} + kx = 0$$

$$\Rightarrow \ddot{x} + \frac{3}{2} \frac{k}{m} x = 0$$

This is required equation of motion.

15.8. (b). The Equations of Motion and First Integrals

Let us consider only the central force, where the potential V is the function of r only, so that the force is always directed along $\mathbf{r}$. Let a single particle move about a fixed centre of force which we assume to be the origin of co-ordinate system. Using polar co-ordinates (r, θ) , the kinetic energy of particle is given by

$$T = \frac{1}{2} \mu (\dot{r}^2 + r^2 \dot{\theta}^2), \quad \mu \text{ being reduced mass.}$$

The potential energy

$$V = V(r).$$

The Lagrangian of the system is given by

$$L = T - V$$

$$= \frac{1}{2} \mu (\dot{r}^2 + r^2 \dot{\theta}^2) - V(r) \quad \dots(1)$$

Lagrange's equation for θ is $\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = 0$... (2)

$\therefore$ From (1)

$$\frac{\partial L}{\partial \theta} = 0 \text{ and } \frac{\partial L}{\partial \dot{\theta}} = \mu r^2 \dot{\theta}$$

$\therefore$ From (2)

so that

$$\frac{d}{dt} (\mu r^2 \dot{\theta}) = 0$$

Integrating, we get

$$\mu r^2 \dot{\theta} = \text{constant} = J \text{ (say)}, \quad \dots(3)$$

where J is first integral (constant of motion) and represents the magnitude of angular momentum.

As μ is constant, equation (2) gives

$$\frac{d}{dt} (r^2 \dot{\theta}) = 0$$

or

$$\frac{d}{dt} \left(\frac{1}{2} r^2 \dot{\theta} \right) = 0,$$

so that

$$\frac{1}{2} r^2 \dot{\theta} = \text{constant} \quad \dots(4)$$

The term $\frac{1}{2} r^2 \dot{\theta}$ represents the *areal velocity*, i.e., the area swept out by the radius vector per unit time. This can be verified from fig. 15-21.

If vector r rotates by an angle $d\theta$ in time dt , the area swept out by r in time dt is $\frac{1}{2} r \cdot (r d\theta) = dA$ (say), so that

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{1}{2} r^2 \dot{\theta}$$

From equation (4);

$$\frac{dA}{dt} = \frac{1}{2} r^2 \dot{\theta} = \text{constant} \quad \dots(5)$$

Equation (1) gives $\frac{\partial L}{\partial \dot{r}} = \mu \dot{r}$

$$\frac{\partial L}{\partial r} = \mu r \dot{\theta}^2 - \frac{\partial V}{\partial r}$$

The Lagrangian equation in terms of r is given by

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{r}} \right) - \frac{\partial L}{\partial r} = 0$$

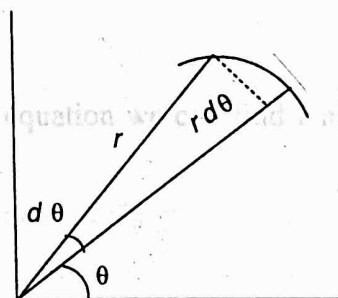


Fig. 15-21

or $\frac{d}{dt}(\mu \dot{r}) - \mu r \dot{\theta}^2 + \frac{\partial V}{\partial r} = 0$... (7)

from (6)

(1) If we represent the force along r by $F(r)$, then we have

$$F(r) = -\frac{\partial V}{\partial r},$$

so that equation (7) can be written as

$$\mu \ddot{r} - \mu r \dot{\theta}^2 = F(r).$$

This is the general equation of motion.

But from equation (3), $\dot{\theta} = \frac{J}{\mu r^2}$.

$$\dot{\theta}^2 = \frac{J^2}{\mu^2 r^4},$$

so that equation (8) gives

$$\mu \ddot{r} - \frac{J^2}{\mu r^3} = F(r).$$

This is second order differential equation in r only

Equation (9) gives

$$\mu \ddot{r} = \frac{J^2}{\mu r^3} + F(r),$$

or

$$\mu \ddot{r} = \frac{J^2}{\mu r^3} - \frac{\partial V}{\partial r} = -\frac{1}{2} \frac{\partial}{\partial r} \left(\frac{J^2}{\mu r^2} \right) - \frac{\partial V}{\partial r}$$

or

$$\mu \ddot{r} = -\frac{\partial}{\partial r} \left(\frac{1}{2} \frac{J^2}{\mu r^2} + V \right). \quad \dots (10)$$

Multiplying both sides of this equation by $\dot{r}$, we get

$$\mu \dot{r} \ddot{r} = -\frac{\partial}{\partial r} \left(\frac{1}{2} \frac{J^2}{\mu r^2} + V \right) \dot{r}$$

or

$$\frac{d}{dt} \left(\frac{1}{2} \mu \dot{r}^2 \right) = -\frac{d}{dt} \left(\frac{1}{2} \frac{J^2}{\mu r^2} + V \right)$$

or

$$\frac{d}{dt} \left(\frac{1}{2} \mu \dot{r}^2 + \frac{1}{2} \frac{J^2}{\mu r^2} + V \right) = 0,$$

$$\frac{1}{2} \mu \dot{r}^2 + \frac{1}{2} \frac{J^2}{\mu r^2} + V = \text{constant}$$

But K.E. = $T = \frac{1}{2} \mu (\dot{r}^2 + r^2 \dot{\theta}^2)$

$$= \frac{1}{2} \mu \left(\dot{r}^2 + \frac{J^2}{\mu^2 r^2} \right)$$

$$= \frac{1}{2} \mu \dot{r}^2 + \frac{1}{2} \frac{J^2}{\mu r^2}$$

and potential energy = V

and total energy $E = T + V = \frac{1}{2} \mu \dot{r}^2 + \frac{1}{2} \frac{J^2}{\mu r^2} + V.$

From (11) and (12), we have

$$\frac{1}{2} \mu \dot{r}^2 + \frac{1}{2} \frac{J^2}{\mu r^2} + V = E = \text{constant},$$

... (13)

i.e., the total energy of the system is constant, i.e., the total energy E , is constant of motion. This is another first integral of motion. This equation (9) represents equation of motion while angular momentum and total energy are constants of motion.

From equation (13), we have

$$\frac{1}{2} \mu \dot{r}^2 = E - \frac{J^2}{2\mu r^2} - V$$

$$\dot{r} = \sqrt{\left[\frac{2}{\mu} \left(E - \frac{J^2}{2\mu r^2} - V \right) \right]}$$

or

$$\frac{dr}{dt} = \sqrt{\left[\frac{2}{\mu} \left(E - \frac{J^2}{2\mu r^2} - V \right) \right]}$$

or

$$\frac{dr}{\sqrt{\left[\frac{2}{\mu} \left(E - \frac{J^2}{2\mu r^2} - V \right) \right]}} = dt \quad \dots(14)$$

Let the initial value of r be r_0 ; then integrating equation (14), we get

$$\int_{r_0}^r \frac{dr}{\sqrt{\left\{ \frac{2}{\mu} \left(E - V - \frac{J^2}{2\mu r^2} \right) \right\}}} = \int_{\theta_0}^{\theta} dt$$

or

$$t = \int_{r_0}^r \frac{dr}{\sqrt{\left\{ \frac{2}{\mu} \left(E - V - \frac{J^2}{2\mu r^2} \right) \right\}}} \quad \dots(15)$$

This equation gives t as a function of r . However, from this equation we can find r as function of t and the constants.

From equation (3), we have

$$d\theta = \frac{J}{\mu r^2} dt$$

If initially $\theta = \theta_0$, then integration of above equation yields

$$\int_{\theta_0}^{\theta} d\theta = \int_0^t \frac{J}{\mu r^2} dt$$

or

$$\theta - \theta_0 = \int_0^t \frac{J}{\mu r^2} dt$$

or

$$\theta = \int_0^t \frac{J}{\mu r^2} dt + \theta_0 \quad \dots(16)$$

This equation gives θ as a function of t .

Equations (15) and (16) are the only integrations to be solved. Therefore, the problems have been reduced to quadratures with four constants of integration E, J, r_0, θ_0 .

15.8 (c) The Differential Equation for the Orbit

Under central force

We have $J = \mu r^2 \dot{\theta} = \text{constant}$

$$\text{and} \quad E = \frac{1}{2} \mu \dot{r}^2 + \frac{J^2}{2\mu r^2} + V = \text{constant} \quad \dots(1)$$

$$\text{Differential equation in } r \text{ is} \quad \mu \ddot{r} - \frac{J^2}{\mu r^3} = F(r) \quad \dots(2)$$

From equation (1)

$$J = \mu r^2 \dot{\theta}$$

or

$$J = \mu r^2 \frac{d\theta}{dt}$$

or

$$J dt = \mu r^2 d\theta \quad \dots(4)$$

The corresponding relation between the derivative relative to t and θ can be written as

$$\frac{d}{dt} = \frac{J}{\mu r^2} \frac{d}{d\theta} \quad \dots(5)$$

The relations (4) and (5) are used to find the differential equation for the orbit from (3). From equation (5) the second derivative w.r. to t is given by

$$\frac{d^2}{dt^2} = \frac{J}{\mu r^2} \frac{d}{d\theta} \left[\frac{J}{\mu r^2} \frac{d}{d\theta} \right]$$

The equation (3) may be written as

$$\mu \frac{J}{\mu r^2} \frac{d}{d\theta} \left\{ \frac{J}{\mu r^2} \frac{dr}{d\theta} \right\} - \frac{J^2}{\mu r^3} = F(r).$$

or

$$\frac{J}{r^2} \frac{d}{d\theta} \left\{ \frac{J}{\mu r^2} \frac{dr}{d\theta} \right\} - \frac{J^2}{\mu r^3} = F(r) \quad \dots(6)$$

To simplify above equation we must remember that

$$\frac{1}{r^2} \frac{dr}{d\theta} = - \frac{d(1/r)}{d\theta} \quad \dots(7)$$

Using (7), equation (6) gives

$$\frac{J^2}{\mu r^2} \frac{d}{d\theta} \left[- \frac{d(1/r)}{d\theta} \right] - \frac{J^2}{\mu r^3} = F(r). \quad \dots(8)$$

Substituting

$$u = \frac{1}{r}.$$

Equation (8) gives

$$- \frac{J^2 u^2}{\mu} \frac{d^2 u}{d\theta^2} - \frac{J^2}{\mu} u^2 = F\left(\frac{1}{u}\right).$$

or

$$\frac{J^2 u^2}{\mu} \left[\frac{d^2 u}{d\theta^2} + u \right] = - F\left(\frac{1}{u}\right). \quad \dots(9)$$

This is the differential equation for the orbit if the force law F is known.

Ex. 33. A particle describes a circular orbit under the influence of an attractive central force directed towards a point on the circle. Show that the force varies as the inverse fifth power of the distance.

(Rohilkhand 2005, 2000, 1996, 87, Meerut 1986, 83, 81)

Solution. The equation of a circle of radius a passing through the origin, in polar co-ordinates, is given by

$$r = 2a \cos \theta.$$

Putting $r = \frac{1}{u}$, we have $u = \frac{\sec \theta}{2a}$... (1)

which gives, $\frac{du}{d\theta} = \frac{\sec \theta \tan \theta}{2a}$

$$\text{and } \frac{d^2u}{d\theta^2} = \frac{\sec \theta \sec^2 \theta + \sec \theta \tan \theta \tan \theta}{2a} \\ = \frac{\sec^3 \theta + \sec \theta \tan^2 \theta}{2a} \quad \dots (2)$$

From equation (9) of § 15.6c, we have

$$\begin{aligned} F\left(\frac{1}{u}\right) &= -\frac{J^2 u^2}{\mu} \left[\frac{d^2u}{d\theta^2} + u \right] \\ &= -\frac{J^2 u^2}{\mu} \left[\frac{\sec^3 \theta + \sec \theta \tan^2 \theta}{2a} + \frac{\sec \theta}{2a} \right] \quad \text{from (1) and (2)} \\ &= -\frac{J^2 u^2}{2a\mu} [\sec^3 \theta + \sec \theta (\tan^2 \theta + 1)] \\ &= -\frac{J^2 u^2}{2a\mu} [\sec^3 \theta + \sec \theta \sec^2 \theta] \\ &= -\frac{J^2 u^2}{2a\mu} \cdot 2 \sec^3 \theta \\ &= -\frac{J^2 u^2}{a\mu} \sec^3 \theta \\ &= -\frac{J^2 u^2}{a\mu} 8a^3 u^3 \quad \text{from (1)} \\ &= -\frac{8J^2 a^2}{\mu} u^5 \\ &= -\frac{8J^2 a^2}{\mu} \cdot \frac{1}{r^5}, \end{aligned}$$

$$\text{i.e. } F(r) \propto \frac{1}{r^5},$$

i.e., the force varies as the inverse fifth power of the distance.

15.8 (d) The Kepler Problem : Inverse Square Law of Force

The inverse square law is most important of all the central force laws. It results in the deduction of Kepler's laws of planetary motion.

The Kepler's laws of planetary motion are :

- (i) All planets move in elliptical orbits having the sun as one focus.
- (ii) The areas swept out by the radius vector of planet relative to the sun in equal times are equal.

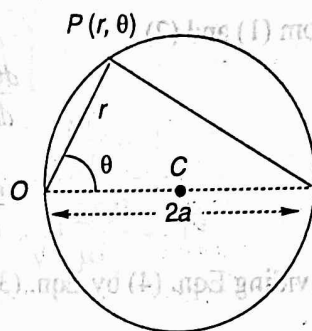


Fig. 15.22

(iii) The square of the period of revolution of any planet about the sun is proportional to the cube of the semi-major axis.

Deduction of Kepler's Laws

Kepler's first law.

Under central force the constants of motion are angular momentum, $J = \mu r^2 \dot{\theta}$

and energy

$$E = \frac{1}{2} \mu \dot{r}^2 + \frac{J^2}{2\mu r^2} + V \quad \dots(1)$$

From (1) and (2)

$$\Rightarrow \frac{d\theta}{dt} = \frac{J}{\mu r^2} \quad \dots(3)$$

and

$$\frac{dr}{dt} = \sqrt{\frac{2}{\mu} \left(E - \frac{J^2}{2\mu r^2} - V \right)} \quad \dots(4)$$

Dividing Eqn. (4) by Eqn. (3), we get

$$\frac{dr}{d\theta} = \frac{\mu r^2}{J} \times \sqrt{\frac{2}{\mu} \left(E - V - \frac{J^2}{2\mu r^2} \right)}$$

or

$$d\theta = \frac{J dr}{\mu r^2 \sqrt{\frac{2}{\mu} \left(E - V - \frac{J^2}{2\mu r^2} \right)}} \quad \dots(5)$$

Under inverse square law of force, we have

$$F(r) = -\frac{k}{r^2},$$

where k is a constant.

But

$$F(r) = -\frac{\partial V}{\partial r}$$

$\therefore$

$$-\frac{\partial V}{\partial r} = -\frac{k}{r^2} \Rightarrow dV = \frac{k}{r^2} dr$$

$$\text{Integrating } V = \int_{\infty}^r \frac{k}{r^2} dr$$

$$\text{or the potential energy } V = -\frac{k}{r} \quad \dots(6)$$

Substituting this value of V in equation (5), we get

$$d\theta = \frac{J dr}{\mu r^2 \sqrt{\frac{2}{\mu} \left(E + \frac{k}{r} - \frac{J^2}{2\mu r^2} \right)}}$$

Integrating, we get

$$\theta = \int \frac{J dr}{\mu r^2 \sqrt{\frac{2}{\mu} \left(E + \frac{k}{r} - \frac{J^2}{2\mu r^2} \right)}} + \theta'$$

where θ' is constant of integration.

Substituting $r = 1/u$, we get

$$\begin{aligned}\theta &= - \int \frac{J du}{\mu \sqrt{\left\{ \frac{2}{\mu} \left(E + ku - \frac{J^2 u^2}{2\mu} \right) \right\}}} + \theta' = \theta' - \int \frac{du}{\sqrt{\left(\frac{2\mu E}{J^2} + \frac{2\mu k u}{J^2} - u^2 \right)}} \\ &= \theta' - \int \frac{du}{\sqrt{\left[\left(\frac{2\mu E}{J^2} + \frac{\mu^2 k^2}{J^4} \right) - \left(u - \frac{\mu k}{J^2} \right)^2 \right]}} \\ &= \theta' - \cos^{-1} \frac{u - \frac{\mu k}{J^2}}{\sqrt{\left(\frac{2\mu E}{J^2} + \frac{\mu^2 k^2}{J^4} \right)}} = \theta' - \cos^{-1} \frac{\frac{uJ^2}{\mu k} - 1}{\sqrt{\left(\frac{2EJ^2}{\mu k^2} + 1 \right)}} \\ \text{or} \quad \frac{\frac{uJ^2}{\mu k} - 1}{\sqrt{\left(\frac{2EJ^2}{\mu k^2} + 1 \right)}} &= \cos(\theta - \theta')\end{aligned}$$

$$\begin{aligned}\text{or} \quad \frac{uJ^2}{\mu k} - 1 &= \sqrt{\left(\frac{2EJ^2}{\mu k^2} + 1 \right)} \cos(\theta - \theta') \\ \text{or} \quad u &= \frac{\mu k}{J^2} \left[1 + \sqrt{\left(\frac{2EJ^2}{\mu k^2} + 1 \right)} \cos(\theta - \theta') \right] \quad \dots(7)\end{aligned}$$

Substituting

$$c = \frac{\mu k}{J^2}$$

and

$$\varepsilon = \sqrt{\left(1 + \frac{2EJ^2}{\mu k^2} \right)}$$

equation (7) gives

$$u = c [1 + \varepsilon \cos(\theta - \theta')]$$

or

$$\frac{1}{r} = c [1 + \varepsilon \cos(\theta - \theta')] \quad \dots(8)$$

which is the equation of the conic with ε as eccentricity and one focus as the origin. Thus the equation of the path of the two-body problem of reduced mass μ is always a conic section, which is the generalisation of *Kepler's first law*.

The nature of the conic depends on the value of eccentricity given by eqn. (8).

If $\varepsilon > 1$, i.e., if $\sqrt{\left(\frac{2EJ^2}{\mu k^2} + 1 \right)} > 1$ or $E > 0$, the conic is *hyperbola*.

If $\varepsilon = 1$, i.e., if $\sqrt{\left(\frac{2EJ^2}{\mu k^2} + 1 \right)} = 1$ or $E = 0$, the conic is a *parabola*.

If $\varepsilon < 1$, i.e., if $\sqrt{\left(\frac{2EJ^2}{\mu k^2} + 1 \right)} < 1$ or $E < 0$, the conic is an *ellipse*.

If $\varepsilon = 0$, i.e., if $\sqrt{\left(\frac{2EJ^2}{\mu k^2} + 1 \right)} = 0$ or $E = -\frac{\mu k^2}{2J^2}$, the conic is a *circle*.

In the case of elliptical orbits, when

$$\theta - \theta' = 0, r = r_1 = \text{perihelion,}$$

then from eqn. (9), we have

$$r_1 = \frac{1}{c(1 + \epsilon)}$$

When $\theta - \theta' = \pi$, $r = r_2$ = aphelion, then eqn. (9) gives

$$r_2 = \frac{1}{c(1 - \epsilon)}$$

The semi-major axis, which is one-half the sum of perihelion r_1 and aphelion r_2 is given by

$$a = \frac{r_1 + r_2}{2} = \frac{1}{2} \left[\frac{1}{c(1 + \epsilon)} + \frac{1}{c(1 - \epsilon)} \right] = \frac{1}{c(1 - \epsilon^2)}$$

Substituting values of c and ϵ from eqn. (8), we get

$$a = \frac{1}{\frac{\mu k}{J^2} \left\{ 1 - \left(1 + \frac{2EJ^2}{\mu k^2} \right) \right\}} = -\frac{k}{2E}$$

or

$$E = -\frac{k}{2a} \quad \dots(10)$$

This shows that in the case of elliptical orbits the total energy depends solely on the major axis.

Deduction of Kepler's II law

From (1) $J = \mu r^2 \dot{\theta} = \text{constant}$

This implies

$$\frac{dA}{dt} = \frac{1}{2} r^2 \dot{\theta} = \text{constant, [refer 15.8 (b)]} \quad \dots(11)$$

which represents the areal velocity, i.e., the area swept out by the radius vector per unit time is constant. This means that the areas swept out by the radius vector in equal times are equal which is Kepler's II law.

Deduction of Kepler's III law. If T is the periodic time of describing the complete orbit, the area of the orbit is given by

$$\begin{aligned} A &= \int_0^T \frac{dA}{dt} dt = \int_0^T \frac{1}{2} r^2 \dot{\theta} dt \\ &= \int_0^T \frac{J}{2\mu} dt \quad (\text{since } J = \mu r^2 \dot{\theta}) \\ &= \frac{JT}{2\mu} \end{aligned} \quad \dots(12)$$

But area of the ellipse $A = \pi ab$,
where a and b are the semi-major and semi-minor axes of the ellipse respectively.

Also

$$b = a\sqrt{1 - \epsilon^2} = a \sqrt{\left(1 - 1 - \frac{2EJ^2}{\mu k^2} \right)} = a \sqrt{\left(-\frac{2EJ^2}{\mu k^2} \right)}$$

But

$$E = -\frac{k}{2a} \text{ from equation (10)}$$

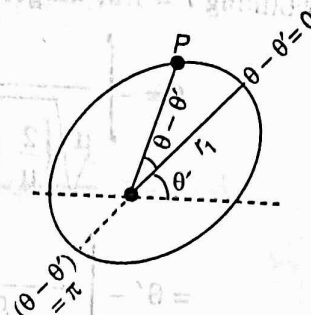


Fig. 15.23

Therefore
$$b = a \sqrt{\left(\frac{kJ^2}{\mu k^2}\right)} = a^{1/2} \sqrt{\left(\frac{J^2}{\mu k}\right)} \quad \dots(14)$$

Substituting value of b in equation (13) we get

$$A = \pi a^{3/2} \sqrt{\left(\frac{J^2}{\mu k}\right)} \quad \dots(15)$$

(1) Comparing (12) and (15), we get

$$\frac{JT}{2\mu} = \pi a^{3/2} \sqrt{\left(\frac{J^2}{\mu k}\right)}$$

Squaring,
$$\frac{J^2 T^2}{4\mu^2} = \pi^2 a^3 \frac{J^2}{\mu k}$$

or
$$T^2 = 4\pi^2 a^3 \frac{\mu}{k} \quad \dots(16)$$

so that

$$T^2 \propto a^3,$$

i.e., the square of the period of revolution of the planet around the sun is proportional to the cube of semi-major axis, which is Kepler's III law.

Ex. 34. A charged particle is moving under the influence of point nucleus. Find the orbit of the particle and the periodic time in the case of an elliptical orbit.

(Agra 1989)

Solution. Let z be the atomic number of the nucleus and e the charge on the electron; then the central force between the nucleus and any electron in orbit is given by

$$F(r) = -\frac{1}{4\pi\epsilon_0} \cdot \frac{ze^2}{r^2} = -\frac{k}{r^2}$$

so that

$$k = \frac{1}{4\pi\epsilon_0} ze^2$$

and

$$\text{eccentricity } \epsilon = \sqrt{\left(1 + \frac{2EJ^2}{\mu k^2}\right)} = \sqrt{\left(1 + \frac{2 \cdot (4\pi\epsilon_0)^2 EJ^2}{\mu z^2 e^4}\right)}$$

But $E = -\frac{k}{2a} = -\frac{1}{4\pi\epsilon_0} \cdot \frac{ze^2}{2a}$; therefore

$$\epsilon = \sqrt{\left\{1 + \frac{2 \left(-4\pi\epsilon_0 \frac{ze^2}{2a}\right) J^2}{\mu z^2 e^4}\right\}} = \sqrt{\left(1 - \frac{4\pi\epsilon_0 J^2}{a\mu ze^2}\right)}$$

i.e.,

$$\epsilon < 1.$$

Therefore equation (5) represents an elliptical orbit, i.e., the path of an electron moving under the influence of point nucleus is an ellipse.

The period T , from equation (15), is given by

$$T^2 = 4\pi^2 a^3 \frac{\mu}{k}$$

Here

$$k = \frac{1}{4\pi\epsilon_0} ze^2.$$

$$\therefore T^2 = 4\pi\epsilon_0 \cdot 4\pi^2 a^3 \cdot \frac{\mu}{ze^2}$$

or

$$T = \frac{2\pi}{e} \cdot \sqrt{\left(\frac{4\pi\epsilon_0 \cdot \mu a^3}{z} \right)}$$

Ex. 35. Use Hamiltonian equations to find the differential equation for planetary motion and prove that areal velocity is constant.
(Rohilkhand 1986, Meerut 1998)

Solution. The Lagrangian L is given by

$$L = T - V = \frac{1}{2} \mu (\dot{r}^2 + r^2 \dot{\theta}^2) + \frac{k}{r} \quad \dots(1)$$

But

$$\frac{\partial L}{\partial \dot{r}} = p_r = \mu \dot{r},$$

and

$$\frac{\partial L}{\partial \dot{\theta}} = p_\theta = \mu r^2 \dot{\theta}$$

The Hamiltonian H is given by

$$H = p_r \dot{r} + p_\theta \dot{\theta} - L$$

$$= p_r \dot{r} + p_\theta \dot{\theta} - \left[\frac{1}{2} \mu (\dot{r}^2 + r^2 \dot{\theta}^2) + \frac{k}{r} \right]$$

$$= p_r \dot{r} + p_\theta \dot{\theta} - \frac{1}{2} \mu \dot{r}^2 - \frac{1}{2} \mu r^2 \dot{\theta}^2 - \frac{k}{r}$$

$$= \frac{p_r^2}{\mu} + \frac{p_\theta^2}{\mu r^2} - \frac{p_r^2}{2\mu} - \frac{p_\theta^2}{2\mu r^2} - \frac{k}{r}$$

$$H = \frac{p_r^2}{2\mu} + \frac{p_\theta^2}{2\mu r^2} - \frac{k}{r} \quad \dots(2)$$

from equation (2)

which yields

$$\frac{\partial H}{\partial p_r} = \frac{p_r}{\mu}, \quad \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{\mu r^2}$$

and

$$\frac{\partial H}{\partial r} = -\frac{p_\theta^2}{\mu r^3} + \frac{k}{r^2}$$

$$\frac{\partial H}{\partial \theta} = 0, \quad \dots(4)$$

According to Hamilton's equations

$$\frac{\partial H}{\partial p_r} = \dot{r}$$

and

$$\frac{\partial H}{\partial r} = -\dot{p}_r$$

From (4) and (5), we get

$$\frac{\partial H}{\partial p_r} = \frac{p_r}{\mu} = \dot{r}$$

and

$$\frac{\partial H}{\partial r} = -\frac{p_\theta^2}{\mu r^3} + \frac{k}{r^2} = -\dot{p}_r \quad \dots(6)$$

As $p_r = \mu \dot{r}$, so that

$$\dot{p}_r = \mu \ddot{r}, \quad \dots(7)$$

From equations (6) and (7), we have

$$-\mu \ddot{r} = -\frac{p_\theta^2}{\mu r^3} + \frac{k}{r^2}$$

or

$$-\mu \ddot{r} = -\frac{(\mu r^2 \dot{\theta})^2}{\mu r^3} + \frac{k}{r^2}$$

(since $p_\theta = \mu r^2 \dot{\theta}$)

or
$$-\mu \ddot{r} = -\mu r^2 \dot{\theta}^2 + \frac{k}{r^2},$$

 i.e.,
$$\ddot{r} = r \dot{\theta}^2 - \frac{k}{\mu r^2} \quad \dots(8)$$

(1) But
$$r^2 \dot{\theta} = h \text{ (say) ,}$$

i.e.,
$$r^2 \frac{d\theta}{dt} = h.$$

(2) Let
$$u = \frac{1}{r} \quad \dots(10)$$

Then
$$\frac{d\theta}{dt} = \frac{h}{r^2} = hu^2$$

and
$$\frac{dr}{dt} = \frac{dr}{d\theta} \cdot \frac{d\theta}{dt} = \frac{d}{d\theta} \left(\frac{1}{u} \right) \cdot hu^2$$

$$= -\frac{1}{u^2} \frac{du}{d\theta} hu^2$$

$$= -h \frac{du}{d\theta}$$

and
$$\frac{d^2r}{dt^2} = -h \frac{d^2u}{d\theta^2} \cdot \frac{d\theta}{dt}$$

or
$$\ddot{r} = -h^2 u^2 \frac{d^2u}{d\theta^2} \quad \dots(12)$$

Substituting values from (9), (10), (11) and (12) in (8), we get

$$-h^2 u^2 \frac{d^2u}{d\theta^2} = \frac{1}{u} h^2 u^4 - \frac{k}{\mu} u^2$$

or
$$-\frac{d^2u}{d\theta^2} = u - \frac{k}{\mu h^2}$$

or
$$\frac{d^2u}{d\theta^2} + u = \frac{k}{\mu h^2} \quad \dots(13)$$

This is the differential equation for planetary motion with $\mu = \frac{1}{r}$.

From equation (4).
$$\frac{\partial H}{\partial \theta} = 0. \quad \dots(14)$$

From Hamilton's equation,

$$\frac{\partial H}{\partial \theta} = -\dot{p}_\theta. \quad \dots(15)$$

Comparing (14) and (15), $\dot{p}_\theta = 0$

or
$$\frac{d}{dt} (p_\theta) = 0$$

i.e.,
$$p_\theta = \text{constant}$$

or
$$\mu r^2 \dot{\theta} = \text{constant}$$

or
$$\frac{1}{2} r^2 \dot{\theta} = \text{constant}$$

(since $\mu = \text{constant}$)

which represents that the areal velocity is constant.

15.8 (c) The Virial Theorem

In order to discuss the Virial theorem let us consider a general system of mass points, out of which i th particle has position vector $\mathbf{r}_i$ and is acted by the applied force $\mathbf{F}_i$. Then the equation of motion of i th particle is given by

$$\mathbf{F}_i = \dot{\mathbf{p}}_i \quad \dots(1)$$

where $\mathbf{p}_i$ is the momentum of i th particle.

Let us introduce a quantity ϕ such that

$$\phi = \sum \mathbf{p}_i \cdot \mathbf{r}_i \quad \dots(2)$$

where the summation is taken over all the particles of the system.

The derivative of ϕ is given by

$$\frac{d\phi}{dt} = \sum_i m_i \dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i + \sum_i \mathbf{F}_i \cdot \mathbf{r}_i \quad \text{from equation (1) and (2)}$$

$$\frac{d\phi}{dt} = \sum_i m_i \dot{\mathbf{r}}_i^2 + \sum_i \mathbf{F}_i \cdot \mathbf{r}_i$$

or

$$\frac{d\phi}{dt} = 2T + \sum_i \mathbf{F}_i \cdot \mathbf{r}_i \quad \dots(3)$$

where T is the kinetic energy of the system.

The time average of equation (3) over a time interval τ is given by

$$\frac{1}{\tau} \int_0^\tau \frac{d\phi}{dt} dt = 2 \overline{T} + \overline{\sum_i \mathbf{F}_i \cdot \mathbf{r}_i} \quad \dots(4)$$

where $\overline{T}$ is the average kinetic energy of the system over time interval τ and $\overline{\sum_i \mathbf{F}_i \cdot \mathbf{r}_i}$ is the average value of $\sum_i \mathbf{F}_i \cdot \mathbf{r}_i$ over the time interval τ .

Equation (4) can be written

$$\frac{1}{\tau} [\phi(\tau) - \phi(0)] = 2\overline{T} + \overline{\sum_i \mathbf{F}_i \cdot \mathbf{r}_i} \quad \dots(5)$$

Let us discuss two types of motion :

(i) **Periodic Motion.** In periodic motion all the co-ordinates repeat after a certain time. If this certain time is chosen to be the time period of motion, which is generally the case, then $\phi(\tau) = \phi(0)$ and hence the L.H.S. of equation (5) vanishes. Then equation (5) becomes

$$2\overline{T} + \overline{\sum_i \mathbf{F}_i \cdot \mathbf{r}_i} = 0.$$

(ii) **Non-periodic Motion.** Let us consider the motion in which the co-ordinates and velocities for all the particles remain finite so that there is an upper bound to ϕ . Then by choosing τ sufficiently long, the L.H.S. of equation (4) can be made as small as we please. Hence for a proper choice of τ , L.H.S. of equation (4) is zero.

Thus, in both the cases L.H.S. of equation (4) is zero and we have

$$2\overline{T} + \overline{\sum_i \mathbf{F}_i \cdot \mathbf{r}_i} = 0$$

or

$$\overline{T} = -\frac{1}{2} \overline{\sum_i \mathbf{F}_i \cdot \mathbf{r}_i} \quad \dots(6)$$

which is the *Virial Theorem*. The R.H.S. equation (6) is called the *Virial of Clausius*.

(Agra 1983)

Solution. For a conservative system, the forces $\mathbf{F}_i$ are derivable from a scalar potential function by taking the gradient of the latter, so that $\mathbf{F} = -\nabla V$; then equation (6) gives.

...(7)

For single particle moving under a central force, the above equation reduces to

... (8)

If the force law varies as r^n , then

...(9)

so that

...(10)

from (9)

Using (10) equation (8) becomes

(11)

In the case of *inverse square law of force* $n = -2$; then equation (11) gives

$$\overline{T} = \frac{-2 + 1}{2} V$$

$$\overline{T} = -\frac{V}{2}$$

or

$$\overline{T} + \frac{V}{2} = 0.$$

or

$$2\overline{T} + V = 0.$$

or

OR

15-8 (f) Scattering in a Central Force Field ; Rutherford Scattering.

15-8 (f) Scattering in a Central Force Field ; Rutherford Scattering.
In order to discuss the scattering of + vely charged particle from the heavy nucleus, we assume ;

- assume ;
- (i) The heavy nucleus and the positively charged particle may be supposed to be point nucleus so that their dimensions are not taken into account.
 - (ii) The nucleus of the atom is so heavy that it is at rest during collision.
 - (iii) The mass of the + vely charged particle may be taken as constant because the velocity of + vely charged particle is very small as compared with the velocity of light.

Let a positively charged particle of charge $z'e$ approach a heavy nucleus N of charge Ze . As both the charges are positive, there will be a force of repulsion between them given by Coulomb's inverse square law. This force of repulsion increases rapidly as the particle gets closer to the nucleus. The positively charged particle of initial velocity v_0 is repelled by the heavy positive nucleus and changes from a straight line to a hyperbola PAQ having one focus at N as shown in fig. 15-24. The asymptotes PO and

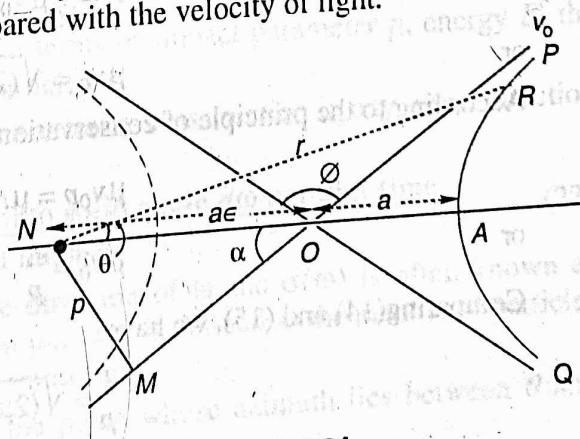


Fig. 15.24

OQ give the initial and final directions of the particle. As the initial and final directions of motion of the particle are not the same, the particle is said to be scattered.

The perpendicular distance of PO from $N = MN = p$. This is the shortest distance from the nucleus to the initial direction of motion of the particle. This is called the *impact parameter*.

Orbit of the particle. The equation of the orbit is given by

$$\frac{1}{r} = c[1 + \epsilon \cos(\theta - \theta')]$$

$$c = \frac{\mu k}{J^2}$$

where

$$\epsilon = \sqrt{\left(1 + \frac{2EJ^2}{\mu k^2}\right)}$$

...(10)

from equation (4) of § 15-6 (d)

$$\text{Here } F = \frac{1}{4\pi\epsilon_0} \frac{zz'e^2}{r^2} = -\frac{k}{r^2}$$

In this case $k = -\frac{zz'e^2}{4\pi\epsilon_0}$, so that $c = -\frac{\mu zz'e^2}{4\pi\epsilon_0 J^2}$, -ve sign indicating that the force is repulsive.

$\therefore$ Equation (1) becomes

$$\frac{1}{r} = -\frac{\mu zz'e^2}{4\pi\epsilon_0 J^2} [1 + \epsilon \cos(\theta - \theta')]$$

If the initial line is set such that $\theta' = 0$, then equation (11) gives

$$\frac{1}{r} = -\frac{\mu zz'e^2}{4\pi\epsilon_0 J^2} [1 + \epsilon \cos \theta]. \quad \dots(12)$$

Substituting value of k in equation (10), we have

$$\epsilon = \sqrt{\left\{1 + \frac{2EJ^2 (4\pi\epsilon_0)^2}{\mu (-zz'e^2)^2}\right\}}$$

or

$$\epsilon = \sqrt{\left\{1 + \frac{(4\pi\epsilon_0)^2 2EJ^2}{\mu z'^2 z^2 e^4}\right\}} \quad \dots(13)$$

If the initial velocity of the particle is v_0 , then we have

$$E = \frac{1}{2} \mu v_0^2$$

or

$$\mu v_0 = \sqrt{(2\mu E)}. \quad \dots(14)$$

According to the principle of conservation of angular momentum, we have

$$\mu v_0 p = \mu r^2 \dot{\theta} = J$$

or

$$\mu v_0 = \frac{J}{p} \quad \dots(15)$$

Comparing (14) and (15), we have

$$\frac{J}{p} = \sqrt{(2\mu E)}$$

or

$$J = p\sqrt{(2\mu E)}. \quad \dots(16)$$

Substituting value of J from (16) in (13), we get

$$\epsilon = \sqrt{1 + \frac{(4\pi\epsilon_0)^2 \cdot 2E \{p\sqrt{(2\mu E)}\}^2}{\mu z^2 z'^2 e^4}} \Rightarrow \epsilon = \sqrt{1 + \left(\frac{8\pi\epsilon_0 E p}{zz'e^2}\right)^2} \quad \dots(17)$$

From this equation it is clear that

$$\epsilon > 1.$$

Therefore equation (12) represents a *hyperbola*.

Angle of scattering. The angle between initial and final directions of the positively charged particle is called angle of scattering, *i.e.*, the angle between the asymptotes is called the angle of scattering. In fig. 15-22, ϕ is the angle of scattering.

From fig. 15-22,

$$\phi + 2\alpha = \pi$$

$$\alpha = \frac{\pi}{2} - \frac{\phi}{2} \quad \dots(18)$$

The asymptotic directions are those for which r is infinite and then $\theta \rightarrow \alpha$, so that equation (12) gives

$$1 + \epsilon \cos \alpha = 0$$

$$\text{or} \quad \cos \alpha = -\frac{1}{\epsilon} \quad \dots(19)$$

Using (18), equation (19) gives

$$\cos \left(\frac{\pi}{2} - \frac{\phi}{2}\right) = -\frac{1}{\epsilon}$$

$$\text{or} \quad \sin \frac{\phi}{2} = -\frac{1}{\epsilon}$$

$$\text{or} \quad \operatorname{cosec} \frac{\phi}{2} = -\epsilon$$

Squaring, we get $\operatorname{cosec}^2 \frac{\phi}{2} = \epsilon^2$

$$\text{or} \quad 1 + \cot^2 \frac{\phi}{2} = 1 + \left(\frac{8\pi\epsilon_0 E p}{zz'e^2}\right)^2$$

$$\text{or} \quad \cot \frac{\phi}{2} = \frac{8\pi\epsilon_0 E p}{zz'e^2} \quad \dots(20)$$

$$\text{or} \quad \phi = 2 \cot^{-1} \frac{8\pi\epsilon_0 E p}{zz'e^2} \quad \dots(21)$$

Above equation gives angle of scattering in terms of impact parameter p , energy E , the charge on the nucleus ze and the charge on the particle $z'e$.

Rutherford-scattering-cross-section. The scattering cross section in any given direction is defined as

$$\sigma(\omega) d\omega = \frac{\text{number of particles scattered into solid angle } d\omega \text{ per unit time}}{\text{incident intensity}} \quad \dots(22)$$

where $d\omega$ is the element of solid angle in the direction of ω and $\sigma(\omega)$ is often known as *differential scattering cross-section*. The incident intensity is defined as the number of particles crossing unit area normal to the incident beam in unit time.

But the differential of solid angle $d\omega$ in the plane whose azimuth lies between θ and $\theta + d\theta$ is $d\omega = \sin \phi d\phi d\theta$.

where scattering is being considered through angle ϕ and $\phi + d\phi$.

The scattering cross-section through angle ϕ in any plane is given by

$$\sigma(\omega) d\omega = \int_{\theta=0}^{2\pi} \sigma(\phi) \sin \phi d\phi d\theta = 2\pi \sigma(\phi) \sin \phi d\phi. \quad \dots(23)$$

If I is incident intensity, the number of particles scattered into solid angle $d\omega$ per unit time

$$= 2\pi I \sigma(\phi) \sin \phi d\phi \quad \dots(24)$$

from (22) and (23),

The cross-section for the particles having a collision parameter between p and $p + dp$ is the area of the radius p and width dp , so that

$$\sigma(p) dp = 2\pi p dp$$

Therefore the number of particles with impact parameter lying between p and $p + dp =$

$$I \cdot 2\pi p dp.$$

As the number of particles scattered into solid angle $d\omega$ lying between ϕ and $\phi + d\phi$ is equal to the number of particles with impact parameter lying between p and $p + dp$, i.e., from (24) and (25), we have

$$I \cdot 2\pi p dp = -2\pi I \sigma(\phi) \sin \phi d\phi. \quad \dots(26)$$

-ve sign is introduced because an increase dp in the impact parameter means less force is exerted on the particle; consequently, scattering angle decreases by an amount $d\phi$.

Eq. (26) gives

$$\sigma(\phi) = -\frac{p dp}{\sin \phi d\phi} \quad \dots(27)$$

Eq. (20) gives

$$p = \frac{zz' e^2}{8\pi\epsilon_0 E} \cot \frac{\phi}{2}.$$

$$dp = -\frac{zz' e^2}{8\pi\epsilon_0 E} \operatorname{cosec}^2 \frac{\phi}{2} \cdot \frac{1}{2} d\phi = -\frac{zz' e^2}{16\pi\epsilon_0 E} \operatorname{cosec}^2 \frac{\phi}{2} d\phi.$$

Substituting values of p and dp in (27) we get

$$\sigma(\phi) = -\frac{\left(\frac{zz' e^2}{8\pi\epsilon_0 E}\right) \cot \frac{\phi}{2} \left(-\frac{zz' e^2}{16\pi\epsilon_0 E}\right) \operatorname{cosec}^2 \frac{\phi}{2} d\phi}{2 \sin \frac{\phi}{2} \cos \frac{\phi}{2} d\phi}$$

$$\therefore \sigma(\phi) = \left(\frac{zz' e^2}{16\pi\epsilon_0 E}\right)^2 \operatorname{cosec}^4 \frac{\phi}{2}. \quad \dots(24)$$

This is well known *Rutherford scattering cross-section*. It shows that the scattering cross-section and hence the number of particles scattered must be proportional to

(i) $\operatorname{cosec}^4 \phi/2$, (ii) the square of the nuclear charge (ze), (iii) the square of the charge on the particle ($z'e$), and (iv) inversely proportional to the square of the initial kinetic energy, i.e. $1/E^2$

15.9. Laboratory and Centre of Mass Systems

In a collision or scattering process at least two particles are involved. To describe such problems there are two coordinate systems.

1. Laboratory Coordinate System (or L-system) : A frame attached with the laboratory is called laboratory coordinate system. It is the usual coordinate system because observations are taken in laboratory coordinate system. This system is abbreviated as *L-system* and in the Rutherford scattering process the target is initially at rest.

2. Centre of Mass Coordinate System (C-system) : A frame attached with the centre of mass of an isolated system of particles is called the centre of mass coordinate system or C-system.

Let us consider an incident particle of mass m_1 having an initial velocity $\mathbf{u}_1$ and the other particle of mass m_2 at rest at O in L -system. Let $\mathbf{r}_1$ and $\mathbf{r}_2$ be the position vectors of m_1 and m_2 relative to a coordinate system fixed in laboratory. Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be velocities of m_1 and m_2 after collision in L -system. As there are no external forces, the linear momentum of the system is conserved.

Then $m_1 \dot{\mathbf{r}}_1 + m_2 \dot{\mathbf{r}}_2 = \mathbf{P}$ (constant)

Initially $\dot{\mathbf{r}}_1 = \mathbf{u}_1$ and $\dot{\mathbf{r}}_2 = 0$

Hence $\mathbf{P} = m_1 \mathbf{u}_1$.

Thus

$$m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2 = \mathbf{P} = m_1 \mathbf{u}_1 \quad \dots(1)$$

Let position vector of centre of mass be $\mathbf{R}$. Then

$$\mathbf{R}_c = \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m_2} \quad \dots(2)$$

Differentiating velocity of centre of mass is given by

$$\mathbf{V}_c = \dot{\mathbf{R}}_c = \frac{m_1 \dot{\mathbf{r}}_1 + m_2 \dot{\mathbf{r}}_2}{m_1 + m_2} = \frac{m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2}{m_1 + m_2}$$

Using (1), we get

$$\mathbf{V}_c = \frac{m_1 \mathbf{u}_1}{m_1 + m_2} \quad \dots(3)$$

Obviously the velocity $\mathbf{V}_c$ of the centre of mass is along the direction of initial velocity $\mathbf{u}_1$ of incident particle.

The centre of mass always lies on the line joining m_1 and m_2 and to an observer fixed at centre of mass, the particle m_1 would appear to be on the line making an angle θ_c with the direction of $\mathbf{V}_c$ and relative to centre of mass the position vector of two masses are [Fig. 15-25(a)].

$$\mathbf{R}_1 = \mathbf{r}_1 - \mathbf{R}_c \text{ and } \mathbf{R}_2 = \mathbf{r}_2 - \mathbf{R}_c \quad \dots(4)$$

These equations give

$$\left. \begin{aligned} \dot{\mathbf{R}}_1 &= \dot{\mathbf{r}}_1 - \dot{\mathbf{R}}_c \text{ or } \mathbf{V}_1 = \mathbf{v}_1 - \mathbf{V}_c \\ \dot{\mathbf{R}}_2 &= \dot{\mathbf{r}}_2 - \dot{\mathbf{R}}_c \text{ or } \mathbf{V}_2 = \mathbf{v}_2 - \mathbf{V}_c \end{aligned} \right\} \quad \dots(5)$$

where $\mathbf{V}_1$ and $\mathbf{V}_2$ are velocities of two particles after collision in C-system. From these equations, we get

$$m_1 \mathbf{V}_1 + m_2 \mathbf{V}_2 = m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2 - (m_1 + m_2) \mathbf{V}_c$$

Using (3), we get

$$m_1 \mathbf{V}_1 + m_2 \mathbf{V}_2 = m_1 \mathbf{u}_1 - (m_1 + m_2) \frac{m_1 \mathbf{u}_1}{m_1 + m_2} = 0 \quad \dots(6)$$

This result indicates that the two particles as seen from centre of mass appear to move in opposite directions in such a way that the net momentum in this system is zero. In other words, the velocity vectors of two particles lie along the line joining the two particles [Fig. 15-25(b)].

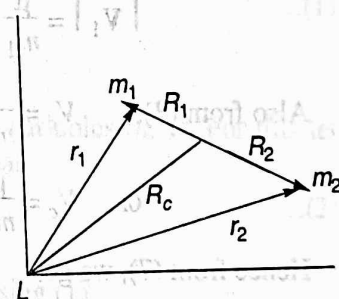


Fig. 15-25(a)



Fig. 15-25(b)

Relations between angles of scattering in Centre of Mass and Laboratory Frames

Let θ_L be the angle of scattering in L -system and θ_c the angle of scattering in C -system [Fig. 15-25(c)].

From Fig.

$$\tan \theta_L = \frac{V_1 \sin \theta_c}{V_1 \cos \theta_c + V_c} \quad \dots(7)$$

In this formula, we have to eliminate V_1 and V_c . For this we have

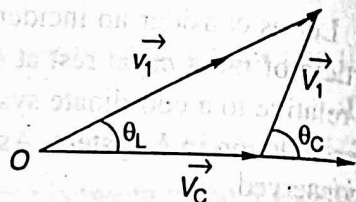


Fig. 15-25(c)

$$\mathbf{R}_1 = \mathbf{r}_1 - \mathbf{R}_c = \mathbf{r}_1 - \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m_2}$$

$$= \frac{m_2}{m_1 + m_2} (\mathbf{r}_1 - \mathbf{r}_2)$$

As reduced mass $\mu = \frac{m_1 m_2}{m_1 + m_2}$ and $\mathbf{r}_1 - \mathbf{r}_2 = \mathbf{r}$, the relative position vector of two particles.

$$\text{Hence } \mathbf{R}_1 = \frac{\mu}{m_1} \mathbf{r}$$

$$\text{or } \dot{\mathbf{R}}_1 = \frac{\mu}{m_1} \dot{\mathbf{r}} \text{ i.e. } \mathbf{V}_1 = \frac{\mu}{m_1} \mathbf{V}$$

where $\mathbf{V}$ is the relative velocity of two particles.

As collision is elastic, the relative velocity of two particles after scattering must be of same magnitude as initial relative velocity, i.e.

$$|\mathbf{V}_1| = \frac{\mu}{m_1} u_1 \quad \dots(8)$$

$$\text{Also from (3), } V_c = \frac{m_1}{m_1 + m_2} u_1$$

$$\text{or } V_c = \frac{\mu}{m_2} u_1$$

Hence from (7), we get

$$\tan \theta_L = \frac{\left(\frac{\mu}{m_1} u_1 \right) \sin \theta_c}{\left(\frac{\mu}{m_1} u_1 \right) \cos \theta_c + \left(\frac{\mu}{m_2} u_1 \right)}$$

$$\text{i.e., } \tan \theta_L = \frac{\sin \theta_c}{\cos \theta_c + \frac{m_1}{m_2}} \quad \dots(9)$$

Case (i). When $m_2 \gg m_1$, $\frac{m_1}{m_2} \ll 1 \Rightarrow \tan \theta_L = \tan \theta_c$ i.e. $\theta_L = \theta_c$

in this case the scattering angles are same in both the systems.

Case (ii). When $m_1 = m_2$ (as in the case of neutron-proton scattering)

$$\tan \theta_L = \frac{\sin \theta_c}{\cos \theta_c + 1} = \tan \frac{\theta_c}{2}$$

$$\text{giving } \theta_L = \frac{\theta_c}{2}$$

Case (iii). When $m_1 \gg m_2$ or $\frac{m_1}{m_2} \gg 1$, the denominator can never be zero. i.e., $\tan \theta_L < \infty$ or $\theta_L < \pi/2$.

This implies that if a heavy particle is incident on a light initially at rest, the heavy particle will not bounce backward as a result of collision.

Relations between scattering cross-sections. As θ_L and θ_C are not equal, the scattering cross-sections will be different when expressed in two different systems. The relation between scattering cross-sections may be found by the fact that the number of particles within a given solid angle must be same in two systems. If $\sigma_C(\theta_C)$ and $\sigma_L(\theta_L)$ are scattering cross-sections in C and L -systems, we have

$$2\pi I \sigma_C(\theta_C) \sin \theta_C d\theta_C = 2\pi I \sigma_L(\theta_L) \sin \theta_L d\theta_L$$
 where I is intensity of incident particles.

$$\text{This gives } \sigma_L(\theta_L) = \sigma_C(\theta_C) \frac{\sin \theta_C}{\sin \theta_L} \cdot \frac{d\theta_C}{d\theta_L} \quad \dots(10)$$

This is the required relation. The value of $\frac{d\theta_C}{d\theta_L}$ may be evaluated using equation (9).

15.10. Legendre Transformations

These transformations provide the change in the basis from one set of coordinates to the other set, i.e., from $(q, \dot{q}, t)$ to (q, p, t) .

Let f be the function of two variables x and y (say) i.e. $f(x, y)$

Then

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = u dx + v dy \quad \dots(1)$$

where

$$u = \frac{\partial f}{\partial x} \text{ and } v = \frac{\partial f}{\partial y}$$

Let us propose to change the basis from (x, y) to independent variables (u, y) . For this let us consider the function g of variable (u, y) i.e. $g = g(u, y)$ such that

$$g(u, y) = f(x, y) - ux \quad \dots(2)$$

$\therefore$

$$\begin{aligned} dg &= df - u dx - x du \\ &= (u dx + v dy) - u dx - x du \text{ using (1)} \\ &= v dy - x du \end{aligned} \quad \dots(3)$$

As

$$g = g(u, y)$$

$\therefore$

$$dg = \frac{\partial g}{\partial u} du + \frac{\partial g}{\partial y} dy \quad \dots(4)$$

Comparing (3) and (4), we get

$$\frac{\partial g}{\partial u} = -x \quad \text{and} \quad \frac{\partial g}{\partial y} = v \quad \dots(5)$$

Clearly these are the converse equations of

$$u = \frac{\partial f}{\partial x} \quad \text{and} \quad v = \frac{\partial f}{\partial y}$$

Thus by the help of equation $g = f - ux$, we may transform the basis from variables (x, y) , to independent variables (u, y) provided x and u satisfy the equations $u = \partial f / \partial x$ and $v = \partial f / \partial y$.

15.11. Canonical Transformations

13.11 (a) **Canonical Transformations.** Sometimes it is convenient to change the old

variables (q_k, p_k) to new variables (Q_k, P_k) . If the transformation are such that the Hamilton's canonical equations

$$\dot{q}_k = \frac{\partial H}{\partial p_k} ; \dot{p}_k = -\frac{\partial H}{\partial q_k} \quad \dots (1)$$

preserve their form in the new variables i.e.,

$$\dot{Q}_k = \frac{\partial H'}{\partial P_k} ; \dot{P}_k = -\frac{\partial H'}{\partial Q_k} \quad \dots (2)$$

H' being the Hamiltonian in the new coordinates, then the transformations are said to be canonical transformations. The new variables Q_k, P_k may be expressed as the functions of old variables

$$\left. \begin{aligned} Q_k &= Q_k(q, p, t) \\ P_k &= P_k(q, p, t) \end{aligned} \right\} \quad \dots (3)$$

and L and L' are the Lagrangian in old and new variables respectively, then

$$H = \sum_k p_k \dot{q}_k - L \text{ and } H' = \sum_k P_k \dot{Q}_k - L' \quad \dots (4)$$

We shall now derive the condition that a transformation from (q, p) basis to (Q, P) basis be canonical. We have seen that the Hamilton's canonical equation may be derived from modified Hamilton's principle in the form

$$\delta \int_{t_1}^{t_2} \left[\sum_k p_k \dot{q}_k - H(p, q) \right] dt = 0 \quad \dots (5)$$

(1). However, if we add to or subtract from the Hamiltonian a total time derivative of a function $F = F(q, p, t)$, we again get the same set of Hamiltonian equations. To show that we add $\frac{dF}{dt}$ to the Hamiltonian H ; so that (5) takes the form

$$\begin{aligned} &= \delta \int_{t_1}^{t_2} \left[\sum_k p_k \dot{q}_k - H(p, q) - \frac{d}{dt} F(p, q, t) \right] dt \\ &= \delta \int_{t_1}^{t_2} \left\{ \sum_k p_k \dot{q}_k - H(p, q) \right\} dt - \delta \int_{t_1}^{t_2} \frac{d}{dt} F(p, q, t) dt \\ &= \delta \int_{t_1}^{t_2} \left\{ \sum_k p_k \dot{q}_k - H(p, q) \right\} dt - \delta \left[F(p, q, t) \right]_{t_1}^{t_2} \\ &= \delta \int_{t_1}^{t_2} \left[\sum_k [p_k \dot{q}_k - H(p, q)] dt - \sum_k \left[\frac{\partial F}{\partial q_k} \delta q_k + \frac{\partial F}{\partial p_k} \delta p_k \right] \right] \\ &= \int_{t_1}^{t_2} \left[\sum_k p_k \dot{q}_k - H(p, q) \right] dt - 0 \quad \dots (6) \end{aligned}$$

since the variations in q_k and p_k vanish at the end points, As the Hamilton's Canonical equations have been derived from Hamilton's principle (5), the conclusion is that *if the two Hamiltonians differ by a total time derivative of a function $F = F(p, q, t)$, the form of canonical equations of motion remains unchanged.*

Canclusively, if we affect a transformation from (q, p) basis to (Q, P) basis which changes the Hamiltonian $H(q, p, t)$ to $H'(Q, P, t)$; the two Hamiltonians would give rise to canonical equations in q, p and Q, P only if these Hamiltonians differ by a total time derivatives of an arbitrary function F : known as the *generating function of transformation* i.e.

$$\delta \int \left[\sum_k p_k \dot{q}_k - H \right] dt = \delta \int \left[\sum_k P_k \dot{Q}_k - H'(P, Q) + \frac{dF}{dt} \right] dt = 0 \quad \dots (7)$$

This equation holds only if the integrands of both sides are same i.e.

$$\sum_k p_k \dot{q}_k - H(p, q) = \sum_k P_k \dot{Q}_k - H'(P, Q) + \frac{dF}{dt} \quad \dots (8)$$

Since the generating function F affects the transformation from old variables (q, p) to new variables (Q, P) , it must be the function of q, p, Q, P and t i.e. in all $(4n + 1)$ coordinates. However, since there exist transformation equations (3), only $(2n + 1)$ coordinates of these $(4n + 1)$ coordinates are independent. Hence F may be expressed as a function of $(2n + 1)$ variables in the forms:

$$F_1(q, Q, t); F_2(q, P, t); F_3(p, Q, t); F_4(p, P, t) = \dots$$

Let us first choose the first form $F_1(q, Q, t)$, then equation (8) can be written as

$$\sum_k p_k \dot{q}_k - H = \sum_k P_k \dot{Q}_k - H' + \frac{dF_1}{dt}(q, Q, t) \quad \dots (9)$$

$$= \sum_k P_k \dot{Q}_k - H' + \sum_k \left(\frac{\partial F_1}{\partial q_k} \dot{q}_k + \frac{\partial F_1}{\partial Q_k} \dot{Q}_k \right) + \frac{\partial F_1}{\partial t}$$

$$\text{i.e. } \sum_k \left(p_k - \frac{\partial F_1}{\partial q_k} \right) \dot{q}_k - \sum_k \left(P_k + \frac{\partial F_1}{\partial Q_k} \right) \dot{Q}_k + \left(H' - H - \frac{\partial F_1}{\partial t} \right) = 0$$

$$\text{i.e., } \sum_k \left(p_k - \frac{\partial F_1}{\partial q_k} \right) \delta q_k - \sum_k \left(P_k + \frac{\partial F_1}{\partial Q_k} \right) \delta Q_k + \left(H' - H - \frac{\partial F_1}{\partial t} \right) \delta t = 0 \quad \dots (10)$$

Since old and new coordinates q_k, Q_k and t may be regarded as independent variables, hence equation (10) holds only if the coefficients of $\delta q_k, \delta Q_k$ and δt vanish separately i.e.

$$p_k = \frac{\partial F_1}{\partial q_k}; P_k = -\frac{\partial F_1}{\partial Q_k} \quad \dots (11)$$

$$\text{and } H' = H + \frac{\partial F_1}{\partial t} \quad \dots (12)$$

Let us now choose the second alternative form for the generating function $F_2(q, P, t)$. Now let us make a transition from $F_1(q, Q, t)$ to $F_2(q, P, t)$ in the following manner. We note that

$$\frac{\partial F_2}{\partial Q_k} = 0 \quad \dots (13)$$

$$\text{and equation (11) shows that } \frac{\partial F_1}{\partial Q_k} + P_k = 0$$

$$\text{Hence we may put } \frac{\partial F_2}{\partial Q_k} = \frac{\partial F_1}{\partial Q_k} + P_k \quad \dots (14)$$

Integrating with respect to variable Q_k ; we get

$$F_2 = F_1 + \sum_k P_k Q_k \quad \dots (15)$$

$$\text{i.e., } F_1 = F_2 - \sum_k P_k Q_k$$

Substituting this in (9); we get

$$\begin{aligned}
\sum_k p_k \dot{q}_k - H &= \sum_k P_k \dot{Q}_k + H' - \frac{d}{dt} \left[F_2(q, P, t) - \sum_k P_k Q_k \right] \\
&= \sum_k P_k \dot{Q}_k - H' - \sum_k \left(\frac{\partial F_2}{\partial q_k} \dot{q}_k + \frac{\partial F_2}{\partial P_k} \dot{P}_k \right) + \frac{\partial F_2}{\partial t} - \sum_k P_k \dot{Q}_k - \sum_k Q_k \dot{P}_k \\
&= -\sum_k \dot{P}_k Q_k - H' + \sum_k \frac{\partial F_2}{\partial q_k} \dot{q}_k + \sum_k \frac{\partial F_2}{\partial P_k} \dot{P}_k + \frac{\partial F_2}{\partial t} \\
\text{i.e., } \sum_k \left(p_k - \frac{\partial F_2}{\partial q_k} \right) \dot{q}_k + \sum_k \left(Q_k - \frac{\partial F_2}{\partial P_k} \right) \dot{P}_k + H' - H - \frac{\partial F_2}{\partial t} &= 0 \\
\text{or } \sum_k \left(p_k - \frac{\partial F_2}{\partial q_k} \right) \delta q_k + \sum_k \left(Q_k - \frac{\partial F_2}{\partial P_k} \right) \delta P_k + \left(H' - H - \frac{\partial F_2}{\partial t} \right) \delta t &= 0. \quad \dots (16)
\end{aligned}$$

As q_k , P_k , and t are independent variables, therefore equation (16) is satisfied if and only if coefficients of δq_k , δP_k and δt vanish separately i.e.

$$p_k = \frac{\partial F_2}{\partial q_k}; Q_k = + \frac{\partial F_2}{\partial P_k} \quad \dots (17)$$

$$\text{and} \quad H' = H + \frac{\partial F_2}{\partial t} \quad \dots (18)$$

Let us now choose third form of generating function $F_3(p, Q, t)$. This may be linked with F_1 as follows.

$$\text{We note that } \frac{\partial F_3}{\partial q_k} = 0 \quad \dots (19)$$

$$\text{From (11) we also have } \frac{\partial F_1}{\partial q_k} - p_k = 0$$

$$\text{Thus we may write } \frac{\partial F_3}{\partial q_k} = \frac{\partial F_1}{\partial q_k} - p_k$$

Integrating with respect to variable q_k ; we get

$$F_3 = F_1 - \sum_k p_k q_k \quad \text{i.e.} \quad F_1 = F_3 + \sum_k p_k q_k \quad \dots (20)$$

Using this equation (9) becomes

$$\begin{aligned}
\sum_k p_k \dot{q}_k - H &= \sum_k P_k \dot{Q}_k - H' + \frac{d}{dt} \left[F_3(p, Q, t) + \sum_k p_k q_k \right] \\
&= \sum_k P_k \dot{Q}_k - H' + \sum_k \left(\frac{\partial F_3}{\partial p_k} \dot{p}_k + \frac{\partial F_3}{\partial Q_k} \dot{Q}_k \right) + \frac{\partial F_3}{\partial t} + \sum_k p_k \dot{q}_k + \sum_k \dot{p}_k q_k \\
\text{or } -\sum_k \left(q_k + \frac{\partial F_3}{\partial p_k} \right) \dot{p}_k - \sum_k \left(P_k + \frac{\partial F_3}{\partial Q_k} \right) \dot{Q}_k + \left(H' - H - \frac{\partial F_3}{\partial t} \right) &= 0 \\
\text{or } -\sum_k \left(q_k + \frac{\partial F_3}{\partial p_k} \right) \delta p_k - \sum_k \left(P_k + \frac{\partial F_3}{\partial Q_k} \right) \delta Q_k + \left(H' - H - \frac{\partial F_3}{\partial t} \right) \delta t &= 0
\end{aligned}$$

As p_k , Q_k and t are independent variables, we must have

$$q_k = -\frac{\partial F_3}{\partial p_k}; P_k = -\frac{\partial F_3}{\partial Q_k} \quad \dots (21)$$

$$\text{and} \quad H' = H + \frac{\partial F_3}{\partial t} \quad \dots (22)$$

Lastly let us choose the generating function $F_4(p, P, t)$. We may link this with F_1 as follows. We note that

$$\frac{\partial F_4}{\partial q_k} = 0, \quad \frac{\partial F_4}{\partial Q_k} = 0 \quad \dots (23)$$

and from (21); we have $\frac{\partial F_3}{\partial Q_k} + P_k = 0$... (24)

$$\therefore \frac{\partial F_4}{\partial Q_k} = \frac{\partial F_3}{\partial Q_k} + P_k$$

Integrating with respect to variable Q_k , we have

$$\begin{aligned} F_4 &= F_3 + \sum_k P_k Q_k \\ &= F_1 - \sum_k p_k q_k + \sum_k P_k Q_k \end{aligned}$$

$$\text{or} \quad F_1(q, Q, t) = F_4(p, P, t) + \sum_k p_k q_k - \sum_k P_k Q_k \quad \dots (25)$$

Substituting this in (9), we get

$$\begin{aligned} \sum_k p_k \dot{q}_k - H &= \sum_k P_k \dot{Q}_k - H' + \frac{d}{dt} \{F_4(p, P, t) + \sum_k p_k q_k - \sum_k P_k Q_k\} \\ &= \sum_k P_k \dot{Q}_k - H' + \sum_k \left(\frac{\partial F_4}{\partial p_k} p_k + \frac{\partial F_4}{\partial P_k} P_k \right) + \frac{\partial F_4}{\partial t} \\ &\quad + \sum_k (p_k \dot{q}_k + \dot{p}_k q_k) - \sum_k (P_k \dot{Q}_k + \dot{P}_k Q_k) \\ \text{i.e.} \quad - \sum_k \left(q_k + \frac{\partial F_4}{\partial p_k} \right) \dot{p}_k + \sum_k \left(Q_k - \frac{\partial F_4}{\partial P_k} \right) \dot{P}_k + \left(H' - H - \frac{\partial F_4}{\partial t} \right) &= 0 \\ \text{i.e.} \quad - \sum_k \left(q_k + \frac{\partial F_4}{\partial p_k} \right) \delta p_k + \sum_k \left(Q_k - \frac{\partial F_4}{\partial P_k} \right) \delta P_k + \left(H' - H - \frac{\partial F_4}{\partial t} \right) \delta t &= 0 \quad \dots (26) \end{aligned}$$

As p_k, P_k and t are independent variables, we must have

$$q_k = - \frac{\partial F_4}{\partial p_k}; \quad Q_k = + \frac{\partial F_4}{\partial P_k} \quad \dots (27)$$

$$H' = H + \frac{\partial F_4}{\partial t} \quad \dots (28)$$

Thus we have obtained canonical transformations for generating functions F_1, F_2, F_3 and F_4 . The results are

$$F_1(q, Q, t); \quad p_k = \frac{\partial F_1}{\partial q_k}; \quad P_k = - \frac{\partial F_1}{\partial Q_k}; \quad H' = H + \frac{\partial F_1}{\partial t} \quad \dots (a)$$

$$F_2(q, P, t) \quad p_k = \frac{\partial F_2}{\partial q_k}; \quad Q_k = \frac{\partial F_2}{\partial P_k}; \quad H' = H + \frac{\partial F_2}{\partial t} \quad \dots (b)$$

$$F_3(p, Q, t) \quad q_k = - \frac{\partial F_3}{\partial p_k}; \quad P_k = - \frac{\partial F_3}{\partial Q_k}; \quad H' = H + \frac{\partial F_3}{\partial t} \quad \dots (c)$$

$$F_4(p, P, t) \quad q_k = - \frac{\partial F_4}{\partial p_k}; \quad Q_k = \frac{\partial F_4}{\partial P_k}; \quad H' = H + \frac{\partial F_4}{\partial t} \quad \dots (d)$$

... (29)

15.11 (b) Conditions for a Transformation to be Canonical :(i) *The transformation from (q, p) , to (Q, P) basis is canonical if the expression*

$$\sum_k (P_k dQ_k - p_k dq_k) \text{ is an exact differential.}$$

Proof. Suppose the generating function does not contain the time explicitly, then from equation (29), $H' = H$. Now for a transformation to be canonical equation (7) must be satisfied i.e.

$$\sum_k P_k \dot{Q}_k - \sum_k p_k \dot{q}_k = \frac{dF}{dt} \quad (\text{Since } H' = H) \quad \dots (30)$$

where $\frac{dF}{dt}$ is the total time derivative of F

$$\text{or} \quad \sum_k P_k dQ_k - \sum_k p_k dq_k = dF$$

As dF is an exact differential of F ; hence for a transformation to be canonical, the expression

$$\sum_k P_k dQ_k - \sum_k p_k dq_k \quad \dots (31)$$

must be an exact differential.

(ii) *Bilinear Invariant as Condition for Canonical Transformations*

A transformation from (q, p) to (Q, P) basis is canonical if the bilinear form

$$\sum_k (\delta p_k dq_k - \delta q_k dp_k) \quad \dots (32)$$

remains invariant

If the transformation from old variables (q, p) to new variables (Q, P) is canonical, then Hamilton's equations may be put as

$$\left. \begin{aligned} \dot{q}_k &= \frac{\partial H}{\partial p_k} & \text{or} & & dq_k &= \frac{\partial H}{\partial p_k} dt \\ \dot{p}_k &= -\frac{\partial H}{\partial q_k} & \text{or} & & dp_k &= -\frac{\partial H}{\partial q_k} dt \end{aligned} \right\} \quad \dots (33)$$

$$\text{and} \quad \left. \begin{aligned} \dot{Q}_k &= \frac{\partial H}{\partial P_k} & \text{or} & & dQ_k &= \frac{\partial H}{\partial P_k} dt \\ \dot{P}_k &= -\frac{\partial H}{\partial Q_k} & \text{or} & & dP_k &= -\frac{\partial H}{\partial Q_k} dt \end{aligned} \right\} \quad \dots (34)$$

Hence for arbitrary $\delta q_k, \delta p_k$; we may write

$$\sum_k \delta p_k \left(dq_k - \frac{\partial H}{\partial p_k} dt \right) - \sum_k \delta q_k \left(dp_k + \frac{\partial H}{\partial q_k} dt \right) = 0$$

$$\text{or} \quad \sum_k (\delta p_k dq_k - \delta q_k dp_k) - \sum_k \left(\frac{\partial H}{\partial q_k} \delta q_k + \frac{\partial H}{\partial p_k} \delta p_k \right) dt = 0$$

$$\sum_k (\delta p_k dq_k - \delta q_k dp_k) - \delta H dt = 0 \quad \dots (35)$$

Similarly, for Q_k, P_k , we get

$$\sum_k (\delta P_k dQ_k - \delta Q_k dP_k) - \delta H dt = 0 \quad \dots (36)$$

Subtracting (36) from (35) and rearranging ; we get

$$\sum_k (\delta p_k dq_k - \delta q_k dp_k) = \sum_k (\delta P_k dQ_k - \delta Q_k dP_k) \dots (37)$$

i.e. $\sum_k (\delta p_k dq_k - \delta q_k dp_k) = \text{invariant form.}$

SOLVED EXAMPLES

Ex. 37. Show that the transformation

$$P = \frac{1}{2} (p^2 + q^2), Q = \tan^{-1} \frac{q}{p} \text{ is canonical}$$

(Kanpur 2007, Rohilkhand 2006, 1992, 79)

Solution. $pdq - PdQ = pdq - \frac{1}{2} (p^2 + q^2) \frac{pdq - qdp}{p^2 + q^2}$

$$= pdq - \frac{1}{2} (pdq - qdp)$$

$$= \frac{1}{2} (pdq + qdp) = d\left(\frac{1}{2} pq\right) = \text{an exact differential}$$

Hence the given transformation is **canonical**.

Ex. 38. Show that the transformation

$$q = \sqrt{2P} \sin Q \text{ and } p = \sqrt{2P} \cos Q \text{ is canonical.}$$

(Meerut 2009, Bombay 2008)

Solution. Given $q = \sqrt{2P} \sin Q$

...(1)

and $p = \sqrt{2P} \cos Q$

...(2)

on differentiating equation (1),

$$dq = \sqrt{2P} \cos Q dQ + (2P)^{-1/2} \sin Q dP$$

$$pdq - PdQ = \sqrt{2P} \cos Q \{ \sqrt{2P} \cos Q dQ + (2P)^{-1/2} \sin Q dP \} - P dQ$$

$$= 2P \cos^2 Q dQ + \sin Q \cos Q dP - P dQ$$

$$= P (2 \cos^2 Q - 1) dQ + \frac{1}{2} \sin 2Q dP$$

$$= P \cos 2Q dQ + \frac{1}{2} \sin 2Q dP$$

$$= d\left[\frac{1}{2} P \sin 2Q\right]$$

This shows that $Pdq - PdQ$ is perfect differential; so given transformation is **canonical**.

Ex. 39. Show that the transformation $Q = \frac{1}{p}$; $P = qp^2$ is canonical using the invariance of bilinear form (Agra 2009)

Solution. We have

$$\delta P dQ - \delta Q dP = (p^2 \delta q + 2qp \delta p) \left(-\frac{1}{p^2} dp\right) - \left(-\frac{1}{p^2} \delta p\right) (p^2 dq + 2qp dp)$$

$$= \delta p dq - \delta q dp - 2qp^{-1} \delta p dp + 2qp^{-1} \delta p dp$$

$$= \delta p dq - \delta q dp = \text{invariance of bilinear form.}$$

Hence the given transformation is canonical

Ex. 40. Show that the transformation

$$Q = \log \left(\frac{1}{q} \sin p \right); P = q \cot p$$

is canonical. Find the generating function $F(q, Q)$.

(Kanpur 2005, Meerut 1993 Rohilkhand 1996, 94)

Solution. We have

$$\begin{aligned} pdq - PdQ &= pdq - q \cot p d \left[\log \left(\frac{1}{q} \sin p \right) \right] \\ &= pdq - q \cot p \cdot \frac{1}{\frac{1}{q} \sin p} d \left(\frac{\sin p}{q} \right) \\ &= pdq - q^2 \frac{\cot p}{\sin p} \frac{q \cos p dp - \sin p dq}{q^2} \\ &= pdq - \cot p (q \cot p dp - dq) \\ &= -q \cot^2 p \cdot dp + (p + \cot p) \cdot dq \\ &= -q (1 - \operatorname{cosec}^2 p) dp + (p + \cot p) dq \\ &= d \{ q (p + \cot p) \} = \text{an exact differential} \end{aligned}$$

Hence the given transformation is canonical. The generating function $F(q, Q)$ satisfies following equations

$$\frac{\partial F}{\partial q} = p; \frac{\partial F}{\partial Q} = -P$$

$$\frac{\partial F}{\partial q} = \sin^{-1}(qe^Q) \quad \dots (1)$$

$$\left[\text{Since } Q = \log \left(\frac{1}{q} \sin p \right) \right]$$

$$\text{i.e. } \sin p = qe^Q$$

$$\frac{\partial F}{\partial Q} = -q \cot p \quad \dots (2)$$

Integrating (1); we get

$$F = \int \sin^{-1}(qe^Q) dq \quad \dots (3) \quad [\text{Choosing constant of integration to be zero for a particular choice of reference system}]$$

To evaluate (3); let $x = \sin^{-1}(qe^Q)$

$$\text{i.e. } \sin x = qe^Q \text{ or } \cos x dx = e^Q dq$$

$$\text{i.e. } dq = \frac{\cos x dx}{e^Q}$$

Hence

$$\begin{aligned} F &= \int \frac{x \cos x dx}{e^Q} = e^{-Q} \left[\sin x \cdot x - \int \sin x dx \right] \\ &= e^{-Q} [x \sin x + \cos x] \end{aligned}$$

$$F(q, Q) = e^{-Q} [qe^Q \sin^{-1}(qe^Q) + \sqrt{1 - q^2 e^{2Q}}]$$

Hence the generating function, $F(q, Q) = e^{-Q} (1 - q^2 e^{2Q})^{1/2} + q \sin^{-1}(qe^Q)$

Ex. 41. Show that the transformation

$Q = \log(1 + \sqrt{q} \cos p)$; $P = 2\sqrt{q}(1 + \sqrt{q} \cos p) \sin p$ is canonical : Find the generating function $F(p, Q)$.
(Agra 2009, Meerut 1993, Rohilkhand 1999, 83)

Solution. Here $pdq - PdQ = pdq - 2(1 + q^{1/2} \cos p) q^{1/2} \sin p \cdot \frac{\cos pdq - 2q \sin pdp}{2q^{1/2}(1 + q^{1/2} \cos p)}$

$$= pdq - \sin p \cos pdq + 2q \sin^2 pdp$$

$$= (p - \frac{1}{2} \sin 2p) dq + q(1 - \cos 2p) dp$$

$$= d\{q(p - \frac{1}{2} \sin 2p)\} = \text{an exact differential}$$

Hence the given transformation is canonical.

We have $F(p, Q) = F_3(p, Q)$ which satisfies the canonical equations

$$\frac{\partial F_3}{\partial p} = -q : \frac{\partial F_3}{\partial Q} = -P$$

Also from $Q = \log(1 + \sqrt{q} \cos p)$; we get
 $q = (e^Q - 1)^2 \sec^2 p$.

Hence $\frac{\partial F_3}{\partial p} = -q = -(e^Q - 1)^2 \sec^2 p$.

Integrating we get

$$F_3 = - \int (e^Q - 1)^2 \sec^2 p dp ; \text{Choosing constant of integration to be zero for a particular choice of reference frame.}$$

$$= -(e^Q - 1)^2 \tan p$$

Hence the generating function for the transformation is

$$F(p, Q) = -(e^Q - 1)^2 \tan p$$

Ex. 42. Find the values of α and β so that the equations

$$Q = q^\alpha \cos \beta p, P = q^\alpha \sin \beta p$$

represent a canonical transformation. What is the form of generating function F_3 for this case?

Solution. The given transformation is canonical if the expression $pdq - PdQ$ is an exact differential,

i.e. $pdq - q^\alpha \sin \beta p d(q^\alpha \cos \beta p) = \text{an exact differential}$

i.e. $pdq - q^\alpha \sin \beta p \{\alpha q^{\alpha-1} \cos \beta p \cdot dq - \beta q^\alpha \sin \beta p \cdot dp\} = \text{an exact differential}$

Now, we know that $Mdx + Ndy$ is an exact differential if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Thus expression (1) i.e.

$(p - \alpha q^{2\alpha-1} \sin \beta p \cos \beta p) dq + \beta q^{2\alpha} \sin^2 \beta p dp$ is an exact differential if

$$\frac{\partial}{\partial p} (p - \alpha q^{2\alpha-1} \sin \beta p \cos \beta p) = \frac{\partial}{\partial q} (\beta q^{2\alpha} \sin^2 \beta p)$$

i.e. $1 - \alpha q^{2\alpha-1} \beta (\cos^2 \beta p - \sin^2 \beta p) = 2\alpha \beta q^{2\alpha-1} \sin^2 \beta p$

i.e. $1 - \alpha \beta q^{2\alpha-1} (1 - 2 \sin^2 \beta p) = 2\alpha \beta q^{2\alpha-1} \sin^2 \beta p$

$$\text{i.e.} \quad 1 - \alpha\beta q^{2\alpha-1} + 2\alpha\beta q^{2\alpha-1} \sin^2 \beta p = 2\alpha\beta q^{2\alpha-1} \sin^2 \beta p$$

$$\text{i.e.} \quad 1 - \alpha\beta q^{2\alpha-1} = 0$$

$$\text{Thereby giving} \quad \alpha\beta q^{2\alpha-1} = 1$$

$$\text{or} \quad \alpha\beta = 1 \quad \text{and} \quad 2\alpha - 1 = 0$$

$$\text{Hence} \quad \alpha = \frac{1}{2} \quad \text{and} \quad \beta = 2$$

$$\text{Thus we have} \quad Q = q^{1/2} \cos 2p, \quad P = q^{1/2} \sin 2p$$

$$\therefore \quad q = Q^2 \sec^2 2p; \quad P = Q \tan 2p$$

If $F_3(p, Q)$ is the generating function for the transformation,

$$\text{then} \quad \frac{\partial F_3}{\partial p} = -q; \quad \frac{\partial F_3}{\partial Q} = -P$$

$$\text{i.e.} \quad \frac{\partial F_3}{\partial p} = -Q^2 \sec^2 2p, \quad \frac{\partial F_3}{\partial Q} = -Q \tan 2p$$

Integrating first of (4) with respect to p ; we get

$$F_3 = - \int Q^2 \sec^2 2p \, dp = -\frac{1}{2} Q^2 \tan 2p$$

$$\text{This yields} \quad \frac{\partial F_3}{\partial Q} = -Q \tan 2p$$

Hence the generating function of the given canonical transformation is

$$F_3(p, Q) = -\frac{1}{2} Q^2 \tan 2p.$$

Problem of Linear Harmonic Oscillator

Ex. 43. Using $F_1 = \frac{1}{2} m \omega q^2 \cot Q$ as a generating function, obtain an expression for the displacement of the linear harmonic oscillator. (Rohilkhand 2000, Meerut 1997).

Solution. Given generating function

$$F_1(q, Q) = \frac{1}{2} m \omega q^2 \cot Q$$

The canonical transformations for $F_1(q, Q)$ give

$$p = \frac{\partial F_1}{\partial q} = m \omega q \cot Q \quad (a)$$

$$P = -\frac{\partial F_1}{\partial Q} = \frac{1}{2} m \omega q^2 \operatorname{cosec}^2 Q \quad (b)$$

From equation (2b); we get

$$q = \sqrt{\left(\frac{2P}{m\omega}\right)} \sin Q$$

and from (2a) and (3):

$$p = \sqrt{(2m\omega P)} \cos Q$$

If k is the force constant of linear harmonic oscillator, then its potential energy is given by

$$V = - \int_0^q F dq = - \int_0^q (-kq) dq = \frac{1}{2} k q^2 \quad (5)$$

∴ The Hamiltonian,

$$H = \frac{1}{2} m \dot{q}^2 + \frac{1}{2} k q^2$$

$$= \frac{p^2}{2m} + \frac{1}{2} m \omega^2 q^2 \quad \left(\text{where } \omega^2 = \frac{k}{m} \right) \quad \dots (6)$$

As the generating function F_1 given by (1) does not contain time explicitly, hence the Hamiltonian H remains unaltered under canonical transformations, i.e.

$$H(p, q) = H(P, Q) \quad \dots (7)$$

Substituting p and q from (3) and (4) in (6); we get

$$H = \omega P \cos^2 Q + \omega P \sin^2 Q = \omega P (\cos^2 Q + \sin^2 Q)$$

$$\text{or} \quad H(P, Q) = \omega P \quad \dots (8)$$

As the new hamiltonian does not contain Q , hence it is cyclic in Q and consequently the corresponding conjugate momentum P is a constant of motion. Obviously,

$$P = \frac{H(P, Q)}{\omega} = \frac{E}{\omega} \quad \dots (9)$$

where $H(P, Q) = E = \text{constant} = \text{total mechanical energy}$.

The other equation of motion is

$$\dot{Q} = \frac{\partial H}{\partial P} = \omega \quad \dots (10)$$

Integration of (10) yields $Q = \omega t + \beta$; β being a constant of integration.

Thus, the displacement of linear harmonic oscillator is given by

$$q = \sqrt{\left(\frac{2E}{m\omega^2} \right)} \sin(\omega t + \beta) \quad \dots (11)$$

15-12. Brackets

15-12 (a) Poisson Brackets. Let u and v be any two arbitrary functions of a set of canonical variables $(q_1, q_2 \dots q_n, p_1, p_2, \dots, p_n)$, then Poisson bracket of u and v is denoted by $(u, v)_{q, p}$ and defined as

$$[u, v]_{q, p} = \sum_k \left(\frac{\partial u}{\partial q_k} \frac{\partial v}{\partial p_k} - \frac{\partial u}{\partial p_k} \frac{\partial v}{\partial q_k} \right) \quad \dots (1)$$

15-12 (b) Properties of Poisson-brackets :

(i) The Poisson-brackets does not obey commutative law, i.e.

$$[u, v]_{q, p} \neq [v, u]_{q, p}$$

Proof. We have

$$[u, v]_{q, p} = \sum_k \left(\frac{\partial u}{\partial q_k} \frac{\partial v}{\partial p_k} - \frac{\partial u}{\partial p_k} \frac{\partial v}{\partial q_k} \right)$$

$$= - \sum_k \left(\frac{\partial v}{\partial q_k} \frac{\partial u}{\partial p_k} - \frac{\partial v}{\partial p_k} \frac{\partial u}{\partial q_k} \right) = - [v, u]_{q, p} \quad \dots (2)$$

Hence

$$[u, v]_{q, p} \neq [v, u]_{q, p}$$

(ii) The Poisson bracket obeys the distributive law, i.e.

$$[u + v, \omega] = [u, \omega] + [v, \omega]$$

Proof. $[u + v, \omega] = \sum_k \left\{ \frac{\partial(u+v)}{\partial q_k} \frac{\partial \omega}{\partial p_k} - \frac{\partial(u+v)}{\partial p_k} \frac{\partial \omega}{\partial q_k} \right\}$

$$= \sum_k \left(\frac{\partial u}{\partial q_k} \frac{\partial \omega}{\partial q_k} - \frac{\partial u}{\partial p_k} \frac{\partial \omega}{\partial q_k} \right) + \sum_k \left(\frac{\partial v}{\partial q_k} \frac{\partial \omega}{\partial p_k} - \frac{\partial v}{\partial p_k} \frac{\partial \omega}{\partial q_k} \right)$$

$$= [u, \omega] + [v, \omega] \quad \dots (3)$$

(iii) $[u v, \omega] = u [v, \omega] + [u, \omega] v$

Proof. $[u v, \omega] = \sum_k \left(\frac{\partial(uv)}{\partial q_k} \frac{\partial \omega}{\partial p_k} - \frac{\partial(uv)}{\partial p_k} \frac{\partial \omega}{\partial q_k} \right)$

$$= \sum_k u \left(\frac{\partial v}{\partial q_k} \frac{\partial \omega}{\partial p_k} - \frac{\partial v}{\partial p_k} \frac{\partial \omega}{\partial q_k} \right) + \sum_k \left(\frac{\partial u}{\partial q_k} \frac{\partial \omega}{\partial p_k} - \frac{\partial u}{\partial p_k} \frac{\partial \omega}{\partial q_k} \right) v$$

$$= u [v, \omega] + [u, \omega] v \quad \dots (4a)$$

Similarly, $[u, v\omega] = [u, v] \omega + v [u, \omega] \quad \dots (4b)$

(iv) **Jacobi-Identity** $[u, [v, \omega]] + [v, [\omega, u]] + [\omega, [u, v]] = 0$

Proof. We have

$$[v, \omega]_{q,p} = \sum_k \left(\frac{\partial v}{\partial q_k} \frac{\partial \omega}{\partial p_k} - \frac{\partial v}{\partial p_k} \frac{\partial \omega}{\partial q_k} \right)$$

$$\therefore [u, [v, \omega]]_{q,p} = \sum_{l,k} \left[\frac{\partial u}{\partial q_l} \frac{\partial}{\partial p_l} \left(\frac{\partial v}{\partial q_k} \frac{\partial \omega}{\partial p_k} - \frac{\partial v}{\partial p_k} \frac{\partial \omega}{\partial q_k} \right) - \frac{\partial u}{\partial p_l} \frac{\partial}{\partial q_l} \left(\frac{\partial v}{\partial q_k} \frac{\partial \omega}{\partial p_k} - \frac{\partial v}{\partial p_k} \frac{\partial \omega}{\partial q_k} \right) \right]$$

$$= \sum_{l,k} \left[\frac{\partial u}{\partial q_l} \left(\frac{\partial^2 v}{\partial p_l \partial q_k} \frac{\partial \omega}{\partial p_k} + \frac{\partial v}{\partial q_k} \frac{\partial^2 \omega}{\partial p_l \partial p_k} - \frac{\partial^2 v}{\partial p_l \partial p_k} \frac{\partial \omega}{\partial q_k} - \frac{\partial v}{\partial p_k} \frac{\partial^2 \omega}{\partial p_l \partial q_k} \right) \right.$$

$$\left. - \frac{\partial u}{\partial p_l} \left(\frac{\partial^2 v}{\partial q_l \partial q_k} \frac{\partial \omega}{\partial p_k} + \frac{\partial v}{\partial q_k} \frac{\partial^2 \omega}{\partial q_l \partial p_k} - \frac{\partial^2 v}{\partial q_l \partial p_k} \frac{\partial \omega}{\partial p_k} - \frac{\partial v}{\partial p_k} \frac{\partial^2 \omega}{\partial q_l \partial q_k} \right) \right]$$

Similarly, writing for $[v, [\omega, u]]$ and $[w, [u, v]]$ and adding we get the result.

(v) $[p_j, p_k] = 0, [q_j, q_k] = 0$ and $[q_j, p_k] = \delta_{jk}$

Proof. $[p_j, p_k] = \sum_l \left(\frac{\partial p_j}{\partial q_l} \frac{\partial p_k}{\partial p_l} - \frac{\partial p_k}{\partial q_l} \frac{\partial p_j}{\partial p_l} \right) = 0 \quad \left[\text{since } \frac{\partial p_j}{\partial q_j} = \frac{\partial p_k}{\partial q_j} = 0 \right]$

Similarly, $[q_j, q_k] = 0$.

and $[q_j, p_k] = \sum_l \left(\frac{\partial q_j}{\partial q_l} \frac{\partial p_k}{\partial p_l} - \frac{\partial q_j}{\partial p_l} \frac{\partial p_k}{\partial q_l} \right)$

$$= \sum_l \frac{\partial p_k}{\partial p_l} = \delta_{jk}$$

(vi) **Poisson Bracket for Linear Momentum $\mathbf{p}$ and Angular Momentum $\mathbf{L}$:**

If $\mathbf{r}$ is position vector and $\mathbf{p}$ the linear momentum of a particle, then angular momentum

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}$$

In terms of cartesian components

$$(\hat{i} L_x + \hat{j} L_y + \hat{k} L_z) = (\hat{i} x + \hat{j} y + \hat{k} z) \times (\hat{i} p_x + \hat{j} p_y + \hat{k} p_z)$$

$$= \hat{i} (y p_z - z p_y) + \hat{j} (z p_x - x p_z) + \hat{k} (x p_y - y p_x)$$

Comparing coefficients of $\hat{i}$, $\hat{j}$ and $\hat{k}$, we get

$$\left. \begin{aligned} L_x &= yp_z - zp_y \\ L_y &= zp_x - xp_z \\ L_z &= xp_y - yp_x \end{aligned} \right\} \quad \dots (1)$$

The Poisson Bracket of functions u and v is

$$[u, v]_{q,p} = \sum_k \left(\frac{\partial u}{\partial q_k} \frac{\partial v}{\partial p_k} - \frac{\partial u}{\partial p_k} \frac{\partial v}{\partial q_k} \right)$$

Here coordinates are $x_1 = x$, $x_2 = y$, $x_3 = z$ and momenta

$$\text{are } p_1 = p_x \quad p_2 = p_y \quad p_3 = p_z.$$

Therefore Poisson Bracket of linear momentum p_x and angular momentum L_x is

$$\begin{aligned} [p_x, L_x]_{x_i, p_i} &= \sum \left\{ \left(\frac{\partial p_x}{\partial x_i} \cdot \frac{\partial L_x}{\partial p_i} \right) - \left(\frac{\partial p_x}{\partial p_i} \frac{\partial L_x}{\partial x_i} \right) \right\} \\ &= \sum_{i=1}^3 \left\{ \frac{\partial p_x}{\partial x_i} \frac{\partial (yp_z - zp_y)}{\partial p_i} - \frac{\partial p_x}{\partial p_i} \frac{\partial (yp_z - zp_y)}{\partial x_i} \right\} \\ &= 0, \quad \left\{ \text{since } \frac{\partial p_x}{\partial x_i} = 0 \text{ and } \frac{\partial p_x}{\partial p_i} \cdot \frac{\partial}{\partial x} (yp_z - zp_y) = 0 \right\} \end{aligned}$$

Now Poisson Bracket of p_x and L_y is

$$\begin{aligned} [p_x, L_y]_{x_i, p_i} &= \sum_{i=1}^3 \left(\frac{\partial p_x}{\partial x_i} \frac{\partial L_y}{\partial p_i} - \frac{\partial p_x}{\partial p_i} \frac{\partial L_y}{\partial x_i} \right) \\ &= \sum_{i=1}^3 \left\{ \frac{\partial p_x}{\partial x_i} \cdot \frac{\partial (zp_x - xp_z)}{\partial p_i} - \frac{\partial p_x}{\partial p_i} \frac{\partial (zp_x - xp_z)}{\partial x_i} \right\} \end{aligned}$$

$$\text{But } \frac{\partial p_x}{\partial x_i} = 0 \text{ and } \frac{\partial p_x}{\partial p_y} = \frac{\partial p_x}{\partial p_z} = 0$$

$$[p_x, L_y]_{x_i, p_i} = - \frac{\partial p_x}{\partial p_x} \frac{\partial}{\partial x} (zp_x - xp_z) = p_z$$

Similarly

$$[p_x, L_z]_{x_i, p_i} = -p_y$$

Thus $[p_x, L_x] = [p_y, L_y] = [p_z, L_z] = 0$; $[p_x, L_y] = p_z$, $[p_x, L_z] = -p_y$;

$$[p_y, L_x] = -p_z, [p_y, L_z] = p_x$$

$$[p_z, L_x] = p_y$$

$$[p_z, L_y] = -p_x$$

15.12 (c) Invariance of Poisson Bracket under Canonical Transformation (C. T.)

Let u and v be any two arbitrary functions of canonical variables (q, p) which are themselves transformed to another set of canonical variables (Q, P) ; then invariance property of Poisson brackets under canonical transformation may analytically be expressed as

$$[u, v]_{q,p} = [u, v]_{Q,P} \quad \dots (1)$$

From definition

$$[u, v]_{q,p} = \sum_k \left(\frac{\partial u}{\partial q_k} \frac{\partial v}{\partial p_k} - \frac{\partial u}{\partial p_k} \frac{\partial v}{\partial q_k} \right) \quad \dots (2)$$

We may further write

$$\left. \begin{aligned} \frac{\partial v}{\partial p_k} &= \sum_l \frac{\partial v}{\partial Q_l} \frac{\partial Q_l}{\partial p_k} + \sum_l \frac{\partial v}{\partial P_l} \frac{\partial P_l}{\partial p_k} \\ \frac{\partial v}{\partial q_k} &= \sum_l \frac{\partial v}{\partial Q_l} \frac{\partial Q_l}{\partial q_k} + \sum_l \frac{\partial v}{\partial P_l} \frac{\partial P_l}{\partial q_k} \end{aligned} \right\} \dots (3)$$

Substituting these in (2); we get

$$\begin{aligned} [u, v]_{q,p} &= \sum_k \sum_l \left[\frac{\partial u}{\partial q_k} \left(\frac{\partial v}{\partial Q_l} \frac{\partial Q_l}{\partial p_k} + \frac{\partial v}{\partial P_l} \frac{\partial P_l}{\partial p_k} \right) - \frac{\partial u}{\partial p_k} \left(\frac{\partial v}{\partial Q_l} \frac{\partial Q_l}{\partial q_k} + \frac{\partial v}{\partial P_l} \frac{\partial P_l}{\partial q_k} \right) \right] \\ &= \sum_l \left(\frac{\partial v}{\partial Q_l} [u, Q_l]_{q,p} + \frac{\partial v}{\partial P_l} [u, P_l]_{q,p} \right) \end{aligned} \dots (4)$$

In order to evaluate Poisson's bracket of $[u, Q_l]$ and $[u, P_l]$, we use the same equation (4). Replacing Q_l for u and u for v , we get

$$\begin{aligned} [Q_l, u]_{q,p} &= \sum_m \frac{\partial u}{\partial Q_m} [Q_l, Q_m] + \sum_m \frac{\partial u}{\partial P_m} [Q_l, P_m]_{q,p} \\ &= \sum_m \frac{\partial u}{\partial P_m} \delta_{lm} \end{aligned} \quad \left[\begin{array}{l} \text{since } [Q_l, Q_m] = 0 \\ [Q_l, P_m] = \delta_{lm} \end{array} \right]$$

Hence $[u, Q_l]_{q,p} = - \frac{\partial u}{\partial P_l} \dots (5)$

Similarly $[u, P_l]_{q,p} = \frac{\partial u}{\partial Q_l} \dots (6)$

Substituting (5) and (6) in (4), we get

$$\begin{aligned} [u, v]_{q,p} &= \sum_l \left(- \frac{\partial v}{\partial Q_l} \frac{\partial u}{\partial P_l} + \frac{\partial v}{\partial P_l} \frac{\partial u}{\partial Q_l} \right) \\ &= \sum_l \left(\frac{\partial u}{\partial Q_l} \frac{\partial v}{\partial P_l} - \frac{\partial u}{\partial P_l} \frac{\partial v}{\partial Q_l} \right) \end{aligned}$$

i.e. $[u, v]_{q,p} = [u, v]_{Q,P}$

Thus under canonical transformations, the Poisson bracket remains invariant.

15.12. (d) Lagrange Brackets

The Lagrange bracket of two dynamical variables $u(q, p)$, $v(q, p)$ is defined

$$\{u, v\}_{q,p} = \sum_j \left(\frac{\partial q_j}{\partial u} \frac{\partial p_j}{\partial v} - \frac{\partial p_j}{\partial u} \frac{\partial q_j}{\partial v} \right) \dots (1)$$

Such expressions are obtained when we find the condition for the invariance of the two Jacobian determinants.

15.12. (e) Properties of Lagrange Brackets

Prop. I. The Lagrange bracket is invariant under canonical transformation.

If we define a $2n$ -dimensional cartesian space formed of position and momentum coordinates $q_1, q_2, \dots, q_n, p_1, p_2, \dots, p_n$ and termed as *phase space*, then Poincare-Theorem states that the integral

$$J_1 = \iint_S \sum_j dq_j dp_j \dots (2)$$

is invariant under canonical transformation.

If u, v be two parameters defining a point on the two dimensional surface S as $q_j = q_j(u, v)$ and $p_j = p_j(u, v)$ then the element of the area $dq_j dp_j$ transforms to the element of area $du dv$ by means of Jacobian

$$dq_j dp_j = \frac{\partial (q_j, p_j)}{\partial (u, v)} du dv \quad \dots (3)$$

where

$$\frac{\partial (q_j, p_j)}{\partial (u, v)} = \begin{vmatrix} \frac{\partial q_j}{\partial u} & \frac{\partial p_j}{\partial u} \\ \frac{\partial q_j}{\partial v} & \frac{\partial p_j}{\partial v} \end{vmatrix} \quad \dots (4)$$

Hence J_1 will be invariant for all canonical co-ordinates if

$$J_1 = \iint_S \sum_j dq_j dp_j = \iint_S \sum_j dQ_j dP_j$$

i.e., If

$$\iint_S \sum_j \frac{\partial (q_j, p_j)}{\partial (u, v)} du dv = \iint_S \sum_j \frac{\partial (Q_j, P_j)}{\partial (u, v)} du dv.$$

The region of integration being arbitrary, the two integrals will be equal if their integrands are identical i.e. if

$$\sum_j \frac{\partial (q_j, p_j)}{\partial (u, v)} = \sum_j \frac{\partial (Q_j, P_j)}{\partial (u, v)} \quad \dots (1)$$

or if

$$\sum_j \begin{vmatrix} \frac{\partial q_j}{\partial u} & \frac{\partial p_j}{\partial u} \\ \frac{\partial q_j}{\partial v} & \frac{\partial p_j}{\partial v} \end{vmatrix} = \sum_j \begin{vmatrix} \frac{\partial Q_j}{\partial u} & \frac{\partial P_j}{\partial u} \\ \frac{\partial Q_j}{\partial v} & \frac{\partial P_j}{\partial v} \end{vmatrix}$$

or if

$$\sum_j \left(\frac{\partial q_j}{\partial u} \frac{\partial p_j}{\partial v} - \frac{\partial p_j}{\partial u} \frac{\partial q_j}{\partial v} \right) = \sum_j \left(\frac{\partial Q_j}{\partial u} \frac{\partial P_j}{\partial v} - \frac{\partial P_j}{\partial u} \frac{\partial Q_j}{\partial v} \right)$$

or if

$$\{u, v\}_{q, p} = \{u, v\}_{Q, P} \text{ in Lagrange brackets notation.}$$

This follows that Lagrange bracket is invariant under canonical transformation, and hence it is immaterial which set of canonical co-ordinate is used, i.e., the subscripts q, p can be avoided in writing the Lagrange bracket.

Prop. II. The Lagrange bracket does not obey the commutative law of algebra i.e., $\{u, v\} = -\{v, u\}$.

We have

$$\begin{aligned} \{u, v\} &= \sum_j \left(\frac{\partial q_j}{\partial u} \frac{\partial p_j}{\partial v} - \frac{\partial p_j}{\partial u} \frac{\partial q_j}{\partial v} \right) \\ &= - \sum_j \left(\frac{\partial p_j}{\partial u} \frac{\partial q_j}{\partial v} - \frac{\partial q_j}{\partial u} \frac{\partial p_j}{\partial v} \right) \\ &= - \sum_j \left(\frac{\partial q_j}{\partial v} \frac{\partial p_j}{\partial u} - \frac{\partial p_j}{\partial v} \frac{\partial q_j}{\partial u} \right) \\ &= -\{v, u\}. \end{aligned}$$

Prop. III. $\{q_j, q_k\} = 0, \{p_j, p_k\} = 0, \{q_j, p_k\} = \delta_{jk}$.

We have

$$\{u, v\} = \sum_j \left(\frac{\partial q_j}{\partial u} \frac{\partial p_j}{\partial v} - \frac{\partial p_j}{\partial u} \frac{\partial q_j}{\partial v} \right)$$

$$\begin{aligned} \therefore \{q_j, q_k\} &= \sum_i \left(\frac{\partial q_i}{\partial q_j} \frac{\partial p_i}{\partial q_k} - \frac{\partial p_i}{\partial q_j} \frac{\partial q_i}{\partial q_k} \right) \\ &= 0 \text{ as } \frac{\partial p_i}{\partial q_k} = 0 = \frac{\partial p_i}{\partial q_j} \end{aligned}$$

Similarly $\{p_j, p_k\} = 0$

and

$$\begin{aligned} \{q_j, p_k\} &= \sum_i \left(\frac{\partial q_i}{\partial q_j} \frac{\partial p_i}{\partial p_k} - \frac{\partial p_i}{\partial q_j} \frac{\partial q_i}{\partial p_k} \right) \\ &= \sum_j \delta_{ij} \delta_{ik} \text{ as } \frac{\partial q_i}{\partial q_j} = \delta_{ij}, \frac{\partial p_i}{\partial q_k} = \delta_{jk} \text{ and } \frac{\partial p_i}{\partial q_j} = 0 = \frac{\partial q_i}{\partial p_k} \\ &= \delta_{jk} \end{aligned}$$

15.12 (f) Relation between the Lagrange and Poisson brackets. If $\{u_l, u_i\}$ is a Lagrange bracket and $[u_l, u_j]$ a Poisson bracket,

then $\sum_{l=1}^{2n} \{u_l, u_i\} [u_l, u_j] = \delta_{ij}$

We have, by definitions of Lagrange and Poisson brackets,

$$\begin{aligned} \sum_{l=1}^{2n} \{u_l, u_i\} [u_l, u_j] &= \sum_{l=1}^{2n} \left[\sum_{k=1}^n \sum_{m=1}^n \left(\frac{\partial q_k}{\partial u_l} \frac{\partial p_k}{\partial u_i} - \frac{\partial p_k}{\partial u_l} \frac{\partial q_k}{\partial u_i} \right) \left(\frac{\partial u_l}{\partial q_m} \frac{\partial u_j}{\partial p_m} - \frac{\partial u_l}{\partial q_m} \frac{\partial u_j}{\partial p_m} \right) \right] \dots (1) \end{aligned}$$

On the right hand side of (1) there are four terms of which the first is

$$\begin{aligned} \sum_{k,m=1}^n \frac{\partial p_k}{\partial u_i} \frac{\partial u_j}{\partial p_m} \sum_{l=1}^{2n} \frac{\partial q_k}{\partial u_l} \frac{\partial u_l}{\partial q_m} &= \sum_{k,m} \frac{\partial p_k}{\partial u_i} \frac{\partial u_j}{\partial p_m} \cdot \frac{\partial q_k}{\partial q_m} \\ &= \sum_{k,m=1}^n \frac{\partial p_k}{\partial u_i} \frac{\partial u_j}{\partial p_m} \cdot \delta_{km} \\ &= \sum_{k,m=1}^n \frac{\partial p_k}{\partial u_i} \frac{\partial u_j}{\partial p_m} \cdot \frac{\partial p_m}{\partial p_k} \text{ as } \delta_{km} = \frac{\partial p_m}{\partial p_k} \\ &= \sum_{k=1}^n \frac{\partial p_k}{\partial u_i} \frac{\partial u_j}{\partial p_k} \end{aligned}$$

The last of the four terms similarly gives $\sum_{k=1}^n \frac{\partial u_j}{\partial q_k} \frac{\partial q_k}{\partial u_i}$ while the remaining two vanish and therefore (1) gives

$$\begin{aligned} \sum_{l=1}^{2n} \{u_l, u_i\} [u_l, u_j] &= \sum_{k=1}^n \left(\frac{\partial u_j}{\partial p_k} \frac{\partial p_k}{\partial u_i} + \frac{\partial u_j}{\partial q_k} \frac{\partial q_k}{\partial u_i} \right) \\ &= \frac{du_j}{du_i} = \delta_{ij}. \end{aligned}$$

15.12 (g) Invariance of Lagrange and Poisson Brackets as Condition for the Transformations to be Canonical. We have already mentioned in the section 15.8 (e) that a transformation from q, p to Q, P is canonical if

$$\sum_i \left(\frac{\partial q_i}{\partial u} \frac{\partial p_i}{\partial v} - \frac{\partial p_i}{\partial u} \frac{\partial q_i}{\partial v} \right) = \sum_j \left(\frac{\partial Q_j}{\partial u} \frac{\partial P_j}{\partial v} - \frac{\partial P_j}{\partial u} \frac{\partial Q_j}{\partial v} \right) \dots (1)$$

the conditions
prevailing as

$$\{u, v\}_{q,p} = \{u, v\}_{Q,P} \dots (3)$$

when $\ell, k = 1, 2, \dots, n$.

$$\{u, v\} = -\{v, u\}. \quad (4)$$

We observe that in order to effect the transformation $(a_1, \dots, a_n) \rightarrow (a_1, \dots, a_{n-1}, 0)$ once

even so

$$dp_j = \left(\sum_{l=1}^n \frac{\partial q_j}{\partial Q_l} dQ_l + \sum_{l=1}^n \frac{\partial q_j}{\partial P_l} dP_l \right)$$

s and equating to R.H.S. of Q_1 etc. on the R.H.S. of (5),

$$\{P_k, P_l\}_{q,p} = 0 \quad \Bigg\} \quad (7)$$

$$\{Q_k, P_l\}_{q,p} = \delta_{kl} \quad \dots (8)$$

$$k, l = 1, 2, \dots, n.$$

Again, expressing the arbitrary parameters u and v as function of a set of canonical variables q_j, p_j ($j = 1, 2, \dots$) i. e. $(q_1, q_2, \dots, q_n, p_1, p_2, \dots, p_n)$ the Poisson bracket of u and v is given by

$$[u, v]_{q,p} = \sum_j \left(\frac{\partial u}{\partial q_j} \frac{\partial v}{\partial p_j} - \frac{\partial u}{\partial p_j} \frac{\partial v}{\partial q_j} \right) \dots (9)$$

$2n$ canonical variables $(q_1, q_2, \dots, q_n,$

$$\left. \begin{aligned} Q_1 &= Q_1(q_1, q_2, \dots, q_n, p_1, p_2, \dots, p_n) \\ &\dots \dots \dots \dots \dots \dots \\ Q_n &= Q_n(q_1, q_2, \dots, q_n, p_1, p_2, \dots, p_n) \\ P_1 &= P_1(q_1, q_2, \dots, q_n, p_1, p_2, \dots, p_n) \\ &\dots \dots \dots \dots \dots \dots \\ P_n &= P_n(q_1, q_2, \dots, q_n, p_1, p_2, \dots, p_n) \end{aligned} \right\} \dots (10)$$

the condition for the transformations (10) to be canonical can be given in terms of Poisson brackets as

$$\left. \begin{aligned} [P_j, P_k]_{q,p} &= 0 \\ [Q_j, Q_k]_{q,p} &= 0 \end{aligned} \right\} \dots (11)$$

$$[Q_j, P_k]_{q,p} = \delta_{jk} \dots (12)$$

where $j, k = 1, 2, \dots, n$.

Ex. 44. Show that the transformation defined by

$$q = \sqrt{2P} \sin Q \quad p = \sqrt{2P} \cos Q$$

is canonical by using Poisson bracket condition.

Solution. To show that the above transformation is canonical, we use the Poisson-bracket condition

$$[Q, Q] = [P, P] = 0 \text{ and } [Q, P] = 1 \dots (1)$$

we have

$$q = \sqrt{2P} \sin Q \dots (2a)$$

$$p = \sqrt{2P} \cos Q \dots (2b)$$

Squaring and adding (2); we get

$$2P = p^2 + q^2 \dots (3)$$

Dividing (2a) by (2b); we get

$$\text{then } \tan Q = \frac{q}{p} \dots (4)$$

From (3)

$$\frac{\partial P}{\partial q} = q, \quad \frac{\partial P}{\partial p} = p \dots (5)$$

From (4)

$$\sec^2 Q \frac{\partial Q}{\partial q} = \frac{1}{p} \text{ and } \sec^2 Q \frac{\partial Q}{\partial p} = -\frac{q}{p^2} \dots (6)$$

Thus

$$\begin{aligned} [Q, P]_{q,p} &= \left[\frac{\partial Q}{\partial q} \frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p} \frac{\partial P}{\partial q} \right] = \left[\frac{1}{p} \cos^2 Q \cdot p + \frac{q}{p^2} \cos^2 Q \cdot q \right] \\ &= \cos^2 Q \left(1 + \frac{q^2}{p^2} \right) = \cos^2 Q (1 + \tan^2 Q) = 1. \end{aligned}$$

Hence the given transformation is canonical.

15.12. (h) Equation of Motion in Poisson-Bracket Form. Let f be a function depending on positions q , momenta p and time t

$$\text{i.e. } f = f(q, p, t) \dots (1)$$

Then total time derivative of f is

$$\frac{df}{dt} = \sum_k \frac{\partial f}{\partial q_k} \dot{q}_k + \sum_k \frac{\partial f}{\partial p_k} \dot{p}_k + \frac{\partial f}{\partial t} \dots (2)$$

If H is the Hamilton of system; then Hamilton's canonical equations are

$$\dot{q}_k = \frac{\partial H}{\partial p_k} \text{ and } \dot{p}_k = -\frac{\partial H}{\partial q_k}$$

using (3); equation (2); becomes

$$\begin{aligned}\frac{df}{dt} &= \sum_k \frac{\partial f}{\partial q_k} \frac{\partial H}{\partial p_k} - \sum_k \frac{\partial f}{\partial p_k} \frac{\partial H}{\partial q_k} + \frac{\partial f}{\partial t} \\ &= \sum_k \left(\frac{\partial f}{\partial q_k} \frac{\partial H}{\partial p_k} - \frac{\partial f}{\partial p_k} \frac{\partial H}{\partial q_k} \right) + \frac{\partial f}{\partial t} = [f, H]_{q,p} + \frac{\partial f}{\partial t} \quad \therefore (4)\end{aligned}$$

This is equation of motion in *Poisson-bracket form*.

If f is an integral (constant) of motion; then $\frac{df}{dt} = 0$; so

$$[f, H] + \frac{\partial f}{\partial t} = 0 \quad \dots (5)$$

If the function f does not depend upon time explicitly, then

$$\frac{\partial f}{\partial t} = 0; \text{ therefore } [f, H] = 0 \quad \dots (6)$$

This means that the function f , not involving explicit time dependence, is an integral of motion if its Poisson bracket with Hamiltonian vanishes.

15.12(a) Hamilton's Equations of Motion in Poisson Bracket Form

Equation of motion of function $F(q, p, t)$ in Poisson's bracket form is

$$\frac{dF}{dt} = [F, H]_{q,p} + \frac{\partial F}{\partial t} \quad \dots (1)$$

If we put q_k and p_k its equation (1) in turn and note that q_k, p_k do not depend on time explicitly, so $\frac{\partial q_k}{\partial t} = 0$ and $\frac{\partial p_k}{\partial t} = 0$, we get

$$\dot{q}_k = [q_k, H] \quad \dots (2)$$

$$\text{and } \dot{p}_k = [p_k, H] \quad \dots (3)$$

Equations (2) and (3) represent Hamilton's canonical equation in terms of Poisson Bracket forms.

Ex. 45. Show that the Poisson bracket of two integrals of motion is itself an integral of motion. (Kerala 2004, Grahwal 1991)

Solution. According to Jacobi identity of three arbitrary functions u, v, ω of variables (q, p) , we have

$$[u, (v, \omega)] + [v, (\omega, u)] + [\omega, (u, v)] = 0 \quad \dots (1)$$

Assuming u and v any two integrals of motion and putting $\omega = H$, we get

$$[u, (v, H)] + [v, (H, u)] + [H, (u, v)] = 0 \quad \dots (2)$$

As u and v are integrals (constant) of motion; we have

$$[v, H] = [H, u] = 0 \quad \dots (3)$$

In view of this, equation (2) yields

$$[H, (u, v)] = 0 \quad \dots (4)$$

From this it follows that

$$[u, v] = \text{constant}$$

Hence Poisson bracket of two integrals of motion is itself an integral of motion. This result is called Poisson's theorem.

15.13. Hamilton-Jacobi Theory

Hamilton Jacobi theory provides the canonical transformation in which all the coordinates, position as well as momenta, are cyclic.

15.13 (a) Hamilton-Jacobi Equation. Consider dynamical system with Hamiltonian $H(q_k, p_k, t)$. The Hamilton's canonical equations

$$\dot{q}_k = \frac{\partial H}{\partial p_k}; \quad \dot{p}_k = -\frac{\partial H}{\partial q_k} \quad \dots (1)$$

If we make a canonical transformation from $q-p$ basis to $Q-P$ basis, thereby changing the Hamiltonian to $H'(Q, P, t)$, then equations of motion in new coordinates take the form

$$\dot{Q}_k = \frac{\partial H'}{\partial P_k}, \quad \dot{P}_k = -\frac{\partial H'}{\partial Q_k}; \quad H' = H + \frac{\partial F}{\partial t} \quad \dots (2)$$

where F is a generating function of the transformation. Now if Q_k and P_k are *cyclic variables*; then

$$\dot{Q}_k = \frac{\partial H'}{\partial P_k} = 0, \quad \dot{P}_k = -\frac{\partial H'}{\partial Q_k} = 0$$

i.e. $Q_k = \text{constant}; P_k = \text{constant} = \alpha_k$ (say) $\dots (3)$

This equation will certainly hold if we assume without loss of generality that new Hamiltonian H' be zero; then from equation (2); we shall have

$$H(q_k, p_k, t) + \frac{\partial F}{\partial t} = 0 \quad \dots (4)$$

For the validity of equation (4): it is convenient to choose F to be of the form $F_2(q, P, t) = S(q, P, t)$ say.

With this form of generating function, we have

$$p_k = \frac{\partial S}{\partial q_k}; \quad Q_k = \frac{\partial S}{\partial P_k} \quad \dots (5)$$

Replacing p_k in equation (4) through (5); we get

$$H\left(q_k, \frac{\partial S}{\partial q_k}, t\right) + \frac{\partial S}{\partial t} = 0 \quad \dots (6a)$$

$$\text{i.e. } H\left(q_1, q_2 \dots q_n, \frac{\partial S}{\partial q_1}, \frac{\partial S}{\partial q_2} \dots \frac{\partial S}{\partial q_n}, t\right) + \frac{\partial S}{\partial t} = 0 \quad \dots (6b)$$

This equation is known as **Hamilton Jacobi equation**. This is a first order partial differential equation in S , involving $(n+1)$ independent variables q_k ($k = 1, 2, \dots n$) and t . The function which is solution of equation (6) is called **Hamilton's Principal Function**,

Physical Significance of Hamilton's Principal Function: Let us determine the total time derivative of $S(q, P, t)$

$$\begin{aligned} \text{i.e. } \frac{dS}{dt} &= \sum_k \frac{\partial S}{\partial q_k} \dot{q}_k + \sum_k \frac{\partial S}{\partial P_k} \dot{P}_k + \frac{\partial S}{\partial t} \\ &= \sum_k p_k \dot{q}_k + \frac{\partial S}{\partial t} \quad \left(\text{Since } \dot{P}_k = 0 \text{ and } p_k = \frac{\partial S}{\partial q_k} \right) \\ &= \sum_k p_k \dot{q}_k - H \quad \left[\text{using (4) with } F \text{ replaced by } S \right] \\ &= L \end{aligned}$$

Thus
$$S = \int L dt \quad \dots (7)$$

This identifies S with Hamilton's principal function.

15.13 (b) Hamilton-Jacobi Equation for Hamilton's Characteristic Function. For conservative system in which the Hamiltonian does not depend on time explicitly, Hamiltonian (H) is constant and is equal to the total energy (E) of the system i.e.

$$H = \alpha = E = \text{total energy} \quad \dots (1)$$

In this case the Hamilton-Jacobi equation may be expressed in

$$H\left(q_k, \frac{\partial S}{\partial q_k}\right) + \frac{\partial S}{\partial t} = 0 \quad \dots (2)$$

In this equation second term involves time while the first term is independent of time, therefore each term must be equal to a constant with opposite signs i.e.

$$H\left(q_k, \frac{\partial S}{\partial q_k}\right) = \alpha_1 \text{ (say) and } \frac{\partial S}{\partial t} = -\alpha_1 \quad \dots (3)$$

Integrating second of (3); we get

$$S = W(q_k, \alpha_k) - \alpha_1 t \quad \dots (4)$$

where $W(q_k, \alpha_k)$ is constant of integration and independent of time. α_k are new constant momenta. The function W thus introduced is called the **Hamilton's characteristic function**. As $\frac{\partial S}{\partial q_k} = \frac{\partial W}{\partial q_k}$; therefore the **Hamilton-Jacobi equation for Hamilton's characteristic function** may be expressed as

$$H\left(q_k, \frac{\partial W}{\partial q_k}\right) + \frac{\partial S}{\partial t} = 0 \quad \dots (5)$$

Physical Significance of Hamilton's Characteristic Function :

$$\text{As } W = W(q_k, \alpha_k) \quad \dots (6)$$

Taking total time derivative i.e.

$$\begin{aligned} \frac{dW}{dt} &= \sum_k \frac{\partial W}{\partial q_k} \dot{q}_k + \sum_k \frac{\partial W}{\partial \alpha_k} \dot{\alpha}_k = \sum_k \frac{\partial W}{\partial q_k} \dot{q}_k \quad (\text{Since } \alpha_k = \text{constant}) \\ &= \sum_k p_k \dot{q}_k \quad \left(\text{Since } \frac{\partial S}{\partial q_k} = \frac{\partial W}{\partial q_k} = p_k\right) \end{aligned} \quad \dots (7)$$

Integrating (7), we get

$$W = \int \sum_k (p_k \dot{q}_k) dt \quad \dots (8)$$

The integral on right hand side is the action; hence **the Hamilton characteristic function W is identified with action :**

Substituting P_k for α_k ; we note that

$$W(q_k, \alpha_k) = W(q_k, P_k) \quad \dots (9)$$

With this form of W , a part of generating function S , a canonical transformation may be generated as follows.

$$p_k = \frac{\partial S}{\partial q_k} = \frac{\partial W}{\partial q_k}; \quad Q_k = \frac{\partial W}{\partial P_k} = \frac{\partial W}{\partial \alpha_k} \quad \dots (10)$$

The condition for determining W is that Hamiltonian H should be equal to α_1 i.e.

$$H(q_k, p_k) = H\left(q_k, \frac{\partial S}{\partial q_k}\right) = H\left(q_k, \frac{\partial W}{\partial q_k}\right) = \alpha_1 \quad \dots (11)$$

As the generating function W is time independent ; the old and new Hamiltonians are the same

$$i.e., \quad H' = H + \frac{\partial W}{\partial t} = H = \alpha_1 \quad \dots (12)$$

Then the new coordinates and momenta are given as

$$\dot{Q}_k = \frac{\partial H'}{\partial P_k} = \begin{cases} 1 & \text{for } k = 1 \\ 0 & \text{for } k \neq 1 \end{cases} \quad \dots (13)$$

$$\text{and} \quad \dot{P}_k = -\frac{\partial H'}{\partial Q_k} = 0 \quad \dots (14)$$

Equation (14) immediately gives

$$P_k = \alpha_k \quad \dots (15)$$

where α_k 's are some constants.

Integration of (13) gives

$$\left. \begin{aligned} Q_1 &= t + \beta_1 = \frac{\partial W}{\partial \alpha_1} \quad \dots (a) \\ \text{and} \quad Q_k &= \beta_k = \frac{\partial W}{\partial \alpha_k}, \quad k \neq 1 \quad \dots (b) \end{aligned} \right\} \quad \dots (16)$$

Thus we note that the time independent Hamilton's characteristic function $W(q_k, P_k = \alpha_k)$ generates a cononical transformation in which all the new coordinates, except one, are cyclic. The only coordinate which is not cyclic is Q_1 given by (16 a) i.e.

$$Q_1 = t + \beta_1 = \frac{\partial W}{\partial \alpha_1} = \frac{\partial W}{\partial E} \quad \dots (17)$$

This relationship shows that the time and energy are canonically conjugate variables

SOLVED EXAMPLES

Ex. 46. Derive Hamilton Jacobi equation for a particle of mass m moving in a force field with its potential given by

$$V = -\frac{K \cos \theta}{r^2} \quad \dots (18) \quad (\text{Rohilkhand 2000})$$

Solution. The kinetic energy of a particle is given by

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \quad \dots (19)$$

The transformation equation relating cartesian coordinates (x, y, z) and spherical coordinates (r, θ, ϕ) are

$$\begin{aligned} x &= r \sin \theta \cos \phi \\ y &= r \sin \theta \sin \phi \\ z &= r \cos \theta \end{aligned}$$

In spherical polar coordinates, equation (1) becomes

$$T = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2)$$

The Lagrangian

$$L = T - V = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2) + \frac{k \cos \theta}{r^2}$$

Now $p_r = \frac{\partial L}{\partial \dot{r}} = m \dot{r}$, $p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2 \dot{\theta}$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2 \sin^2 \theta \dot{\phi}$$

Hamiltonian

$$H = \sum_k p_k \dot{q}_k - L$$

$$= p_r \dot{r} + p_\theta \dot{\theta} + p_\phi \dot{\phi} - \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2) - \frac{K \cos \theta}{r^2}$$

$$= p_r \left(\frac{p_r}{m} \right) + p_\theta \left(\frac{p_\theta}{mr^2} \right) + p_\phi \left(\frac{p_\phi}{mr^2 \sin^2 \theta} \right)$$

$$- \frac{1}{2} m \left\{ \frac{p_r^2}{m^2} + \frac{p_\theta^2}{m^2 r^2} + \frac{p_\phi^2}{m^2 r^2 \sin^2 \theta} \right\} - \frac{K \cos \theta}{r^2}$$

$$= \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{p_\phi^2}{2mr^2 \sin^2 \theta} - \frac{K \cos \theta}{r^2} \quad \dots (1)$$

Writing $p_r = \frac{\partial S}{\partial r}$, $p_\theta = \frac{\partial S}{\partial \theta}$ and $p_\phi = \frac{\partial S}{\partial \phi}$ equation (1) becomes

$$H = \frac{1}{2m} \left\{ \left(\frac{\partial S}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial S}{\partial \theta} \right)^2 + \frac{1}{r^2 \sin^2 \theta} \left(\frac{\partial S}{\partial \phi} \right)^2 \right\} - \frac{K \cos \theta}{r^2} \quad \dots (2)$$

Hamilton Jacobi equation is

$$H \left(r, \theta, \phi, \frac{\partial S}{\partial r}, \frac{\partial S}{\partial \theta}, \frac{\partial S}{\partial \phi} \right) + \frac{\partial S}{\partial t} = 0 \quad \dots (3)$$

Using (2) above equation becomes

$$\frac{1}{2m} \left\{ \left(\frac{\partial S}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial S}{\partial \theta} \right)^2 + \frac{1}{r^2 \sin^2 \theta} \left(\frac{\partial S}{\partial \phi} \right)^2 \right\} - \frac{K \cos \theta}{r^2} + \frac{\partial S}{\partial t} = 0 \quad \dots (4)$$

This is required Hamilton-Jacobi equation.

Ex. 47. Use Hamilton-Jacobi equation to solve the problem of one dimensional linear harmonic oscillator. (Rohilkhand 2001, 98, 91; Meerut 1992, 82)

Solution. For linear Harmonic oscillator ; potential energy $V(q) = \frac{1}{2} kq^2$ and hence

$$\text{Hamiltonian } H = \frac{p^2}{2m} + V(q) = \frac{p^2}{2m} + \frac{1}{2} kq^2 \quad \dots (1)$$

The Hamilton-Jacobi equation for Hamilton's principal function is

$$\left[\frac{1}{2m} \left(\frac{\partial S}{\partial q} \right)^2 + V(q) \right] + \frac{\partial S}{\partial t} = 0 \quad \dots (2)$$

The first term in bracket is function of q only, while the second term is function of t only, therefore each term must be equal to the same constant with opposite signs.

$$\text{i.e.} \quad \left. \begin{aligned} \frac{1}{2m} \left(\frac{\partial S}{\partial q} \right)^2 + V(q) &= \alpha \\ \frac{\partial S}{\partial t} &= -\alpha \end{aligned} \right\} \quad \dots (3)$$

and

$$\frac{\partial S}{\partial t} = -\alpha$$

Then we have

$$S = W(q, \alpha) - \alpha t, \quad \dots (4)$$

W being a constant of integration.

As $\frac{\partial S}{\partial q} = \frac{\partial W}{\partial q}$; therefore the Hamilton-Jacobi equation for Hamilton's characteristic function W takes the form

$$\frac{1}{2m} \left(\frac{\partial W}{\partial q} \right)^2 + V(q) = \alpha \quad \dots (5)$$

$$\text{this gives } \frac{\partial W}{\partial q} = \sqrt{2m \left\{ \alpha - V(q) \right\}} \quad \dots (6)$$

Integrating above expression, we get

$$W = \int \left[2m \left\{ \alpha - V(q) \right\} \right]^{1/2} dq \quad \dots (7)$$

$$\text{Now } p = \frac{\partial S}{\partial q} = \frac{\partial W}{\partial q} = \left[2m \left\{ \alpha - V(q) \right\} \right]^{1/2} \quad \dots (8)$$

$$\begin{aligned} \text{and } \beta &= \frac{\partial S}{\partial \alpha} = \frac{\partial W}{\partial \alpha} - t \\ &= \frac{\partial}{\partial \alpha} \int \left[2m \left\{ \alpha - V(q) \right\} \right]^{1/2} dq - t \\ \therefore \beta + t &= \sqrt{\frac{m}{2}} \int \frac{dq}{[\alpha - V(q)]^{1/2}} \quad \dots (9) \end{aligned}$$

So far the treatment is general and holds for any problem characterised by one-degree of freedom.

For harmonic oscillator $V(q) = \frac{1}{2} kq^2$;

$$\text{therefore } \beta + t = \sqrt{\frac{m}{2}} \int \frac{dq}{\left[\alpha - \frac{1}{2} kq^2 \right]^{1/2}} = \sqrt{\left(\frac{m}{k} \right)} \int \frac{dq}{\left[\frac{2\alpha}{k} - q^2 \right]^{1/2}}$$

Integrating we get

$$\begin{aligned} \beta + t &= \sqrt{\frac{m}{k}} \cos^{-1} \left[q \sqrt{\frac{k}{2\alpha}} \right] \\ \text{Using } \omega &= \sqrt{\frac{k}{m}}; \text{ we immediately get} \\ q &= \sqrt{\frac{2\alpha}{k}} \cos [\omega(t + \beta)] \quad \dots (10) \end{aligned}$$

Now the only problem is to evaluate α and β using initial conditions.

We note that $\frac{\partial S}{\partial t} = -\alpha$ and also $\frac{\partial S}{\partial t} + H = 0$

Thus we get $-\alpha + H = 0$ or $H = \alpha = \text{total energy} = E$

The initial conditions are at $t = 0$; $q = q_0$ (say) and $p = 0$; then from (5) we have

$$\frac{p^2}{2m} + V(q) = \alpha \quad \left(\text{Since } p = \frac{\partial W}{\partial q} \right)$$

As $p = 0$ at $t = 0$, we get $\alpha = \left[V(q) \right]_{t=0} = \frac{1}{2} k q_0^2 = \frac{1}{2} m \omega^2 q_0^2$

Thus $\alpha = \frac{1}{2} k q_0^2 = \frac{1}{2} m \omega^2 q_0^2 = \text{total energy } E$... (11)

This equation implies that $\sqrt{\frac{2\alpha}{k}} = q_0$; hence equation (10) takes the form

$$q = q_0 \cos \omega(t + \beta) \quad \dots (12)$$

Now using $q = q_0$ at $t = 0$; we get $\beta = 0$.

Hence the displacement equation for harmonic oscillator becomes

$$q = q_0 \cos \omega t \quad \dots (13)$$

Now momentum p is given by

$$p = m \dot{q} = -m\omega q_0 \sin \omega t \quad \dots (14)$$

The Hamilton's principal function

$$\begin{aligned} S &= W(q, \alpha) - \alpha t \\ &= \int [2m(\alpha - \frac{1}{2} k q^2)]^{1/2} dq - \frac{1}{2} m \omega^2 q_0^2 t \quad \text{[using (7) and (11)]} \end{aligned}$$

$$= m\omega \int (q_0^2 - q^2)^{1/2} dq - \frac{1}{2} m \omega^2 q_0^2 t \quad \text{[using (11) again]}$$

$$\text{i.e., } S = m\omega^2 q_0^2 \int (\sin^2 \omega t - \frac{1}{2}) dt \text{ using (13)} \quad \dots (15)$$

Equations (13), (14) and (15) give the required values of q , p and S .

Ex. 48. A mass m is released from rest at height h . Solve the equation of motion for this system using Hamilton-Jacobi theory. (Rohilkhand 1995)

Solution. Taking earth as reference level for zero potential energy, we have

$$V(q) = mgz \quad \dots (1)$$

Here the generalised coordinate q is z . Proceeding exactly as in above example upto equation (9); we have

$$p = \frac{\partial W}{\partial z} = [2m\{\alpha - V(q)\}]^{1/2} \quad \dots (2)$$

$$\text{and } \beta + t = \sqrt{\left(\frac{m}{2}\right)} \int \frac{dq}{[\alpha - V(q)]^{1/2}} \quad \dots (3)$$

$\therefore$ The Hamiltonian of system is

$$H = \frac{p^2}{2m} + mgz = E = \alpha \quad \dots (4)$$

$$\begin{aligned} \text{From; (3)} \quad \beta + t &= \sqrt{\left(\frac{m}{2}\right)} \int \frac{dz}{[E - mgz]^{1/2}} = \sqrt{\frac{m}{2}} \cdot \frac{2}{mg} (E - mgz)^{1/2} \\ &= \frac{1}{g} \sqrt{\frac{2}{m}} (E - mgz)^{1/2} \quad \dots (5) \end{aligned}$$

Thus $(\beta + t)^2 = \frac{1}{g^2} \cdot \frac{2}{m} (E - mgz)$

Solving for z ; we get

$$z = -\frac{1}{2} g (\beta + t)^2 + \frac{E}{mg}$$

Initial conditions are

At $t = 0$, $z = h$, $v = \dot{z} = 0$ and so $p = m\dot{z} = 0$

Using (6); equations (4) and (6) yield

$$E = mgh; h = -\frac{1}{2} g \beta^2 + \frac{E}{mg}$$

Solving these equations; we get $\beta = 0$

Now $p = \frac{\partial W}{\partial z} = m\dot{z} = mgt$ using (6)

Thus we have from (6) and (8)
$$\left. \begin{aligned} z &= h - \frac{1}{2} g t^2 \\ p &= mgt \end{aligned} \right\}$$

These are required equations with $S = W - E t$

where
$$W = \int \sqrt{2m} \cdot (E - mgz)^{1/2} dz$$

15.14 (c) Separation of Variables. Hamilton-Jacobi equation may be used as a powerful technique for solving problems if it is possible to separate the variables. Consider a case in which the Hamiltonian is a constant of motion (it may or may not be the total energy). The Hamilton-Jacobi equation is

$$H\left(q_k, \frac{\partial S}{\partial q_k}\right) + \frac{\partial S}{\partial t} = 0 \quad \dots (1)$$

In this equation the first term is a function of q_k only and second term is a function of t only; therefore each term must be equal to same constant with opposite sign i.e.

$$\left. \begin{aligned} H\left(q_k, \frac{\partial S}{\partial q_k}\right) &= \alpha \text{ (say)} & \dots (a) \\ \text{and} \quad \frac{\partial S}{\partial t} &= -\alpha & \dots (b) \end{aligned} \right\}$$

Integration of above equation yields a solution of the form

$$S = W(q_k, \alpha) - \alpha t$$

$W(q_k, \alpha)$ being constant of integration

Now $\frac{\partial S}{\partial q_k} = \frac{\partial W}{\partial q_k}$; hence we may write (2a) as

$$H\left(q_k, \frac{\partial W}{\partial q_k}\right) = \alpha \quad \dots (4)$$

This is *Hamilton-Jacobi equation for Hamilton's characteristic function* W .

The Hamilton-Jacobi equation is said to be separable if a solution of the form

$$W = W_1(q_1, \alpha_1, \alpha_2, \dots, \alpha_n) + W_2(q_2, \alpha_1, \alpha_2 \dots \alpha_n) + \dots + W_n(q_n, \alpha_1, \alpha_2 \dots \alpha_n) \quad \dots (5)$$

splits the Hamilton-Jacobi equation into n -equations of the form

$$H_k \left(q_k, \frac{\partial W_k}{\partial q_k}, \alpha_1, \dots, \alpha_n \right) = \alpha_k \quad (k=1, 2, \dots, n) \quad \dots (6)$$

This may be solved for $\frac{\partial W}{\partial q_k}$ and expressed as

$$\frac{\partial W_k}{\partial q_k} = \frac{\partial}{\partial q_k} W_k (q_k, \alpha_1, \dots, \alpha_n) \quad \dots (7)$$

Integrating we get

$$W_k = \int \frac{\partial W_k}{\partial q_k} dq_k \quad \dots (8)$$

and then the Hamilton's principal function S is given by

$$S = \sum_{k=1}^n W_k (q_k, \alpha_1, \alpha_2, \dots, \alpha_n) - \alpha t \quad \dots (9)$$

Finally we get

$$p_k = \frac{\partial S}{\partial q_k}; \quad \beta_k = \frac{\partial S}{\partial \alpha_k} \quad \dots (10)$$

and

$$\beta + t = \frac{\partial S}{\partial \alpha_k} = \frac{\partial W}{\partial E} \quad (\text{for } \alpha = E)$$

It may be noted that when Hamilton-Jacobi equation is separated then in general W_k will contain only one q_k but each of them may contain all α_k 's. Let us illustrate the method by following example :

Ex. 49. Use Hamilton-Jacobi equation to discuss the motion of a massible particle in a plane under a central force.

Solution. The Hamiltonian of the particle of mass m reduced mass μ in a plane under a central force in polar coordinates is given by

$$\begin{aligned} H &= \frac{1}{2} \mu (\dot{r}^2 + r^2 \dot{\theta}^2) + V(r) \\ &= \frac{1}{2\mu} \left[p_r^2 + \frac{p_\theta^2}{r^2} \right] + V(r) \quad \dots (1) \end{aligned}$$

$$(\text{since } p_r = \mu \dot{r}, \dot{p}_\theta = \mu r^2 \dot{\theta})$$

As the Hamiltonian does not contain time; hence it is a constant of motion and is equal to total energy E . The Hamilton-Jacobi equation then becomes

$$\frac{\partial S}{\partial t} + \frac{1}{2\mu} \left[\left(\frac{\partial S}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial S}{\partial \theta} \right)^2 \right] + V(r) = 0 \quad \dots (2)$$

The Hamilton's principal function S is given by

$$S = W(r, \theta, \alpha_k) - Et \quad \dots (3)$$

As $\frac{\partial S}{\partial r} = \frac{\partial W}{\partial r}, \frac{\partial S}{\partial \theta} = \frac{\partial W}{\partial \theta}$; therefore the Hamilton-Jacobi equation for Hamilton's characteristic function W becomes

$$\frac{1}{2\mu} \left[\left(\frac{\partial W}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial W}{\partial \theta} \right)^2 \right] + V(r) = E \quad \dots (4)$$

Let us assume that the solution of above equation exists in the form.

$$W = W_r(r) + W_\theta(\theta) \quad \dots (5)$$

where W_r is the function of r only and W_θ is the function of θ only.

Using (5); equation (4) becomes

$$\left(\frac{\partial W_r}{\partial r}\right)^2 + \frac{1}{r^2} \left(\frac{\partial W_\theta}{\partial \theta}\right)^2 = 2\mu \{E - V(r)\}$$

This may be put as

$$\left(\frac{\partial W_\theta}{\partial \theta}\right)^2 = r^2 \left[2\mu \{E - V(r)\} - \left(\frac{\partial W_r}{\partial r}\right)^2 \right] \quad \dots (6)$$

As left hand side is a function of θ only and right hand side is a function of r only; therefore each side must be equal to same constant α_θ^2 (say), i.e.

$$\left(\frac{\partial W_\theta}{\partial \theta}\right)^2 = \alpha_\theta^2 \quad \dots (7a)$$

$$\text{and} \quad r^2 [2\mu \{E - V(r)\}] - \left(\frac{\partial W_r}{\partial r}\right)^2 = \alpha_\theta^2 \quad \dots (7b)$$

Equation (7a) yields $\frac{\partial W_\theta}{\partial \theta} = \alpha_\theta$ or on integration

$$W_\theta = \alpha_\theta \cdot \theta \quad \dots (8)$$

Now from (7b); $\frac{\partial W_r}{\partial r} = \left[2\mu (E - V) - \frac{\alpha_\theta^2}{r^2} \right]^{1/2}$, on integration

$$W_r = \int \left[2\mu (E - V) - \frac{\alpha_\theta^2}{r^2} \right]^{1/2} dr \quad \dots (9)$$

$\therefore$ Characteristic function

$$W = W_r + W_\theta = \int \left[2\mu (E - V) - \frac{\alpha_\theta^2}{r^2} \right]^{1/2} dr + \alpha_\theta \theta \quad \dots (10)$$

and finally Hamilton's principal function

$$S = \int \left[2\mu (E - V) - \frac{\alpha_\theta^2}{r^2} \right]^{1/2} dr + \alpha_\theta \theta - Et \quad \dots (11)$$

Now we have

$$\beta_\theta = \frac{\partial S}{\partial \alpha_\theta} = \frac{\partial W}{\partial \alpha_\theta} = \theta - \int \frac{\alpha_\theta dr}{r^2 \sqrt{\left[2\mu (E - V) - \frac{\alpha_\theta^2}{r^2} \right]}}$$

$$\text{i.e.} \quad \theta = \beta_\theta + \int \frac{\alpha_\theta dr}{r^2 \sqrt{\left[2\mu (E - V) - \frac{\alpha_\theta^2}{r^2} \right]}}$$

Identifying $\beta_\theta = \theta_0$ and changing variable from r to u through the relation $u = \frac{1}{r}$; we get

$$\theta = \theta_0 - \int \frac{du}{\sqrt{\left[\frac{2\mu}{\alpha_\theta^2} (E - V) - u^2 \right]}} \quad \dots (12)$$

This equation gives required equation of orbit.

Also we have

$$\beta + t = \frac{\partial W}{\partial E} = \frac{\partial W_r}{\partial E} = \int \frac{\mu dr}{\sqrt{[2\mu(E - V) - \frac{\alpha^2}{r^2}]}} \quad \dots (13)$$

This equation gives r as a function of time t .

15.13 (d) Action and Angle Variable. Hamilton-Jacobi theory provides a powerful tool to study the motion of periodic systems. In this the old variables (q, p) are transformed to new variables (J, ω) known as action and angle variables by the use of time-independent generating function.

Consider a canonical transformation generated by the function $S(q, \alpha, t)$ given by

$$S(q, \alpha, t) = W(q, \alpha_k) - \alpha t \quad \dots (1)$$

$(k = 1, 2, \dots, n)$

where $W(q, \alpha_k)$ has been obtained by the method of separation of variables and $\alpha = H$. Now, from the transformation property of S ; we have

$$p_k = \frac{\partial S}{\partial q_k} = \frac{\partial Q_k}{\partial q_k} \quad \dots (2)$$

Let us define the quantities J_k by

$$J_k = \oint p_k dq_k \text{ or } J_k = \oint \frac{\partial W_k}{\partial q_k} dq_k \quad \dots (3)$$

Since q_k is the only variable of integration, it does not appear in the final result and each J_k thus a function of n constants of integrations $\alpha_1, \alpha_2, \dots, \alpha_n$ which appear in the solution of Hamilton-Jacobi equation. Since each set of canonical conjugate variables (q_k, p_k) is independent, therefore various J_k 's are independent. Then we may write n equations (for $k = 1, 2, \dots, n$)

$$J_k = J_k(\alpha_1, \alpha_2, \dots, \alpha_n) \quad \dots (4)$$

We may solve above equation for α_k 's in terms of $J_1, J_2, \dots, J_n$. Thus

$$\left. \begin{aligned} \alpha_k &= \alpha_k(J_1, J_2, \dots, J_n) \\ \text{and } H &= H(J) \end{aligned} \right\} \quad \dots (5)$$

Then $W = W(q_k, \alpha_k)$ may be expressed as

$$W(q_k, J_k) = W_1(q_1, J_1, \dots, J_n) + W_2(q_2, J_1, \dots, J_n) + \dots + W_n(q_n, J_1, \dots, J_n)$$

This function may be used as a generating function for a canonical transformation from (q, p) to (ω, J) and expressed by the relations

$$\omega_k = \frac{\partial W}{\partial J_k}(q, J); \quad p_k = \frac{\partial W}{\partial q_k} \quad \dots (6)$$

The quantities J and ω are the action and angle variables respectively.

When action-angle variables (J, ω) are used, the Hamiltonian contains the action variables. Moreover the generating function W is time independent, therefore the new and old Hamiltonians are the same, i.e. $H' = H = H(J)$. and then the new canonical equations of motion become

$$\dot{J}_k = -\frac{\partial H}{\partial \omega_k} = 0 \quad \dot{\omega}_k = \frac{\partial H}{\partial J_k} = \text{constant} = \nu_k(J_k) \quad \dots (7)$$

$$\text{i.e. } J_k = \text{constant}; \quad \omega_k = \nu_k t + \beta_k \quad \dots (8)$$

where β_k 's are constants of integration. The constants v_k will depend on J_k through the partial derivatives $\frac{\partial H}{\partial J_k}$. Let us now see the physical significance of v_k . For this purpose let us first find a change in a given angle variable (say ω_1) due to a complete cycle of variation in any one coordinate say q_k with all other coordinates assumed to be fixed, i.e. change in ω_1 ,

$$\Delta \omega_1 = \oint \left(\frac{\partial \omega_1}{\partial q_k} dq_k + \frac{\partial \omega_1}{\partial p_k} dp_k \right) \quad \dots (9)$$

$$= \oint \left[\frac{\partial}{\partial q_k} \left(\frac{\partial W}{\partial J_1} \right) dq_k + \frac{\partial}{\partial p_k} \left(\frac{\partial W}{\partial J_1} \right) dp_k \right] \quad \left(\text{since } \omega_1 = \frac{\partial W}{\partial J_1} \right)$$

$$= \oint \left[\frac{\partial^2 W}{\partial q_k \partial J_1} dq_k + \frac{\partial^2 W}{\partial p_k \partial J_1} dp_k \right]$$

$$= \oint \left[\frac{\partial}{\partial J_1} \left(\frac{\partial W}{\partial q_k} \right) dq_k + \frac{\partial}{\partial J_1} \left(\frac{\partial W}{\partial p_k} \right) dp_k \right]$$

$$= \left[\frac{\partial}{\partial J_1} \oint \frac{\partial W}{\partial q_k} dq_k \right] \quad \left(\text{since } \frac{\partial W}{\partial p_k} = 0 \right)$$

$$= \left[\frac{\partial}{\partial J_1} \oint p_k dq_k \right] = \frac{\partial J_k}{\partial J_1} \quad \dots (10)$$

i.e. $\Delta \omega_1 = \begin{cases} 1 & \text{if } k = 1 \\ 0 & \text{if } k = 2, 3, \dots, n \end{cases}$

In other words, ω_1 increases by unit whenever q_1 goes through a complete cycle and remains unchanged due to variation in any other coordinate. Now from (8); we have

$$\Delta \omega_1 = v_1 \Delta t \quad (\text{for } k = 1)$$

where Δt is the time required for q_1 to go through a complete cycle once. In that case $\Delta \omega_1 = 1$. Hence,

$$v_1 \Delta t = 1 \quad \text{or} \quad v_1 = \frac{1}{\Delta t} = \frac{1}{\tau} \quad \dots (11)$$

where $\tau = \Delta t$ is the periodic time; thus v_1 must be interpreted as the frequency of motion. With analogy of v_1 , the frequency of motion associated with periodic motion of q_k is given by

$$v_k = \frac{\partial H}{\partial J_k} \quad \dots (12)$$

Using equation (12) we may determine the frequencies of the periodic motion directly without finding the changes of coordinates with time.

For periodic motion; we note that

$$\dot{Q} = \frac{\partial H}{\partial P} = \frac{\partial H}{\partial \alpha} = 1$$

$$\therefore Q = t + \beta' = \frac{\partial S}{\partial \alpha}(q, \alpha)$$

Hence $\tau = \Delta t = \oint d \left(\frac{\partial S}{\partial \alpha} \right) = \oint \left\{ \frac{\partial^2 S}{\partial \alpha^2} d\alpha + \frac{\partial^2 S}{\partial q \partial \alpha} dq \right\}$

$$\begin{aligned}
 &= \oint \frac{\partial}{\partial \alpha} \frac{\partial S}{\partial q} dq \\
 &= \frac{\partial}{\partial \alpha} \oint \frac{\partial S}{\partial q} dq = \oint p dq \quad (\text{since } d\alpha = 0) \\
 \text{i.e.} \quad \tau &= \frac{\partial J}{\partial \alpha} \quad \dots (13)
 \end{aligned}$$

This equation is used to determine periodic time τ directly.

Ex. 50. Determine the frequency of the harmonic oscillator of mass m , force constant k , for which Hamiltonian is given by

$$H = \frac{p^2}{2m} + \frac{1}{2} kq^2$$

(q, p) being position and momentum coordinates of the oscillator.. (Rohilkhand 2005, 1999,86)

Solution. The Hamilton-Jacobi equation for harmonic oscillator is

$$\begin{aligned}
 H\left(q, \frac{\partial S}{\partial q}, t\right) + \frac{\partial S}{\partial t} &= 0 \\
 \frac{1}{2m} \left(\frac{\partial S}{\partial q}\right)^2 + \frac{1}{2} kq^2 + \frac{\partial S}{\partial t} &= 0 \quad \dots (1)
 \end{aligned}$$

(replacing p by $\frac{\partial S}{\partial q}$)

Let its solution be

$$S(q, \alpha, t) = W(q, \alpha) - \alpha t \quad \dots (2)$$

$$\text{with } H = \alpha \quad \text{and} \quad \frac{\partial S}{\partial t} = -\alpha \quad \dots (3)$$

Now the Hamilton-Jacobi equation for W is

$$\frac{1}{2m} \left(\frac{\partial W}{\partial q}\right)^2 + \frac{1}{2} kq^2 = \alpha.$$

This yields

$$\frac{\partial W}{\partial q} = \sqrt{mk} \sqrt{\left(\frac{2\alpha}{k}\right) - q^2} \quad \dots (4)$$

$$\text{and hence} \quad W = \sqrt{mk} \int \sqrt{\left(\frac{2\alpha}{k} - q^2\right)} dq \quad \dots (5)$$

$\therefore$ Action variable

$$\begin{aligned}
 J &= \oint p dq = \oint \frac{\partial W}{\partial q} dq \\
 &= \sqrt{mk} \oint \sqrt{\left(\frac{2\alpha}{k} - q^2\right)} dq \quad \dots (6)
 \end{aligned}$$

[using (4)]

$$= \sqrt{2\alpha m} \oint \left[1 - \frac{kq^2}{2\alpha}\right]^{1/2} dq$$

Substituting

$$q = \sqrt{\frac{2\alpha}{k}} \sin \phi;$$

then

$$J = 2\alpha \sqrt{\frac{m}{k}} \int_0^{2\pi} \cos^2 \phi \, d\phi = 2\pi\alpha \sqrt{\frac{m}{k}}$$

This yields $H = \alpha = \frac{1}{2\pi} \sqrt{\frac{k}{m}}$

$$\therefore \text{The frequency } \nu = \frac{\partial H}{\partial J} = \frac{1}{2\pi} \sqrt{\frac{k}{m}}$$

Ex. 51. Determine the frequency of Kepler problem for a particle in an inverse square central force field.

Solution. The kinetic energy and potential energy of the particle under central force field in spherical polar coordinates are

$$T = \frac{1}{2} m [\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2]$$

$$V = - \int F \, dr = \int \left(-\frac{k}{r^2} \right) dr = -\frac{k}{r}$$

Hamiltonian $H = \sum_k p_k \dot{q}_k - L$

$$= p_r \dot{r} + p_\theta \dot{\theta} + p_\phi \dot{\phi} - L$$

Now Lagrangian,

$$L = T - V = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2) + \frac{k}{r}$$

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}, \quad p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2 \dot{\theta}$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2 \sin^2 \theta \dot{\phi}$$

$$\therefore H = \frac{1}{2m} \left[p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 \sin^2 \theta} \right] - \frac{k}{r} \quad \dots (1)$$

The Hamilton-Jacobi equation $H + \frac{\partial S}{\partial t} = 0$ is obtained on replacing p_k by $\frac{\partial S}{\partial q_k}$

i.e.,

$$\left[p_r = \frac{\partial S}{\partial r}, p_\theta = \frac{\partial S}{\partial \theta}, p_\phi = \frac{\partial S}{\partial \phi} \right]$$

$$\frac{1}{2m} \left[\left(\frac{\partial S}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial S}{\partial \theta} \right)^2 + \frac{1}{r^2 \sin^2 \theta} \left(\frac{\partial S}{\partial \phi} \right)^2 \right] - \frac{k}{r} + \frac{\partial S}{\partial t} = 0 \quad \dots (2)$$

As the explicit time dependence is involved only in the last term, we may write the solution of (2) in the form by the method of separation of variable as

$$S(r, \theta, \phi, \alpha, t) = W(r, \theta, \phi, \alpha) - \alpha t \quad \dots (3)$$

where α is a constant of integration. With this form of S , the Hamilton-Jacobi equation for W becomes

$$\frac{1}{2m} \left[\left(\frac{\partial W}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial W}{\partial \theta} \right)^2 + \frac{1}{r^2 \sin^2 \theta} \left(\frac{\partial W}{\partial \phi} \right)^2 \right] - \frac{k}{r} = \alpha = E \quad \dots (4)$$

where $\alpha = H =$ total energy E of the conservative system.

The solution of W may be expressed as

$$W = W_r(r) + W_\theta(\theta) + W_\phi(\phi) \quad \dots (5)$$

Then equation (4) becomes

$$\left(\frac{\partial W_r}{\partial r}\right)^2 + \frac{1}{r^2} \left(\frac{\partial W_\theta}{\partial \theta}\right)^2 + \frac{1}{r^2 \sin^2 \theta} \left(\frac{\partial W_\phi}{\partial \phi}\right)^2 = 2m \left[E + \frac{k}{r}\right] \quad \dots (6)$$

$$\text{or } r^2 \sin^2 \theta \left[2m \left(E + \frac{k}{r}\right) - \left(\frac{\partial W_r}{\partial r}\right)^2 - \frac{1}{r^2} \left(\frac{\partial W_\theta}{\partial \theta}\right)^2 \right] = \left(\frac{\partial W_\phi}{\partial \phi}\right)^2 \quad \dots (7)$$

In this equation left hand side is a function of r and θ while right hand side is a function of ϕ only ; therefore each side must be equal to constant say α_ϕ^2 , i.e.

$$\frac{\partial W_\phi}{\partial \phi} = \alpha_\phi = \text{constant} = p_\phi \quad \dots (8)$$

$$\text{and } \left(\frac{\partial W_\theta}{\partial \theta}\right)^2 + \frac{\alpha_\phi^2}{\sin^2 \theta} = 2mr^2 \left(E + \frac{k}{r}\right) - r^2 \left(\frac{\partial W_r}{\partial r}\right)^2 \quad \dots (9)$$

By the same reasoning, each side of equation (9) must be equal to a constant say α_θ^2 , i.e.

$$\left(\frac{\partial W_\theta}{\partial \theta}\right)^2 + \frac{\alpha_\phi^2}{\sin^2 \theta} = \alpha_\theta^2 \quad \dots (10)$$

$$\text{and } \left(\frac{\partial W_r}{\partial r}\right)^2 + \frac{\alpha_\theta^2}{r^2} = 2m \left(E + \frac{k}{r}\right) \quad \dots (11)$$

From (10) and (11) ; we get

$$p_\theta = \frac{\partial W_\theta}{\partial \theta} = [\alpha_\theta^2 - \alpha_\phi^2 \operatorname{cosec}^2 \theta]^{1/2} \quad \dots (12)$$

$$p_r = \frac{\partial W_r}{\partial r} = \left[2m \left(E + \frac{k}{r}\right) - \frac{\alpha_\theta^2}{r^2} \right]^{1/2} \quad \dots (13)$$

Now the value of W is obtained as

$$W = \int (p_r dr + p_\theta d\theta + p_\phi d\phi) \quad \dots (14)$$

In order to find frequency we are interested in the evaluation of action and angle variables. The action variables are J_r , J_θ and J_ϕ such that

$$J_r = \oint p_r dr = \oint \frac{\partial W_r}{\partial r} dr = \oint \left[2m \left(E + \frac{k}{r}\right) - \frac{\alpha_\theta^2}{r^2} \right]^{1/2} dr \quad \dots (15)$$

$$J_\theta = \oint p_\theta d\theta = \oint \frac{\partial W_\theta}{\partial \theta} d\theta = \oint \left[\alpha_\theta^2 - \frac{\alpha_\phi^2}{\sin^2 \theta} \right]^{1/2} d\theta \quad \dots (16)$$

$$\text{and } J_\phi = \oint p_\phi d\phi = \oint \frac{\partial W_\phi}{\partial \phi} d\phi = \oint \alpha_\phi d\phi \quad \dots (17)$$

The integral (17) may be evaluated easily since $\alpha_\phi = \text{constant}$ and ϕ varies from 0 to 2π .

$$\text{Hence } J_\phi = \int_0^{2\pi} \alpha_\phi d\phi = 2\pi \alpha_\phi \quad \dots (18)$$

To obtain integral (16), we note that equation (10) gives

$$p_\theta^2 + \frac{p_\phi^2}{\sin^2 \theta} = \alpha_\theta^2 \quad \dots (19)$$

The Hamiltonian in plane polar coordinates is given by

$$H = \frac{1}{2m} \left(p_r^2 + \frac{p_\phi^2}{r^2} \right) - \frac{k}{r} \quad \dots (20)$$

where p is the total angular momentum.

Comparing this equation with equation (1), we note that α_θ must be identified with the total angular momentum p

$$\text{i.e.} \quad \alpha_\theta = p \quad \dots (21)$$

Now kinetic energy expression may be expressed as

$$T = \frac{1}{2} (p_r \dot{r} + p_\theta \dot{\theta} + p_\phi \dot{\phi}) = \frac{1}{2} (p_r \dot{r} + p \dot{\psi}) \quad \dots (22)$$

where (r, θ, ϕ) are spherical polar coordinates and (r, ψ) are plane polar coordinates. Hence

$$p_\theta \dot{\theta} + p_\phi \dot{\phi} = p \dot{\psi} \quad \dots (23)$$

$$\text{which gives} \quad p_\theta d\theta = p d\psi - p_\phi d\phi \quad \dots (24)$$

$$\begin{aligned} \therefore J_\theta &= \oint p_\theta d\theta = \oint (p d\psi - p_\phi d\phi) \\ &= \oint p d\psi - \oint p_\phi d\phi = 2\pi p - 2\pi p_\phi = 2\pi (p - p_\phi) \end{aligned}$$

$$\text{or} \quad J_\theta = 2\pi (\alpha_\theta - \alpha_\phi) = 2\pi \alpha_\theta - J_\phi \quad \dots (25)$$

Lastly action variable

$$\begin{aligned} J_r &= \oint \left[2m \left(E + \frac{k}{r} \right) - \frac{(J_\theta + J_\phi)^2}{4\pi^2 r^2} \right]^{1/2} dr \\ &= \oint \left[2m E + \frac{2mk}{r} - \frac{(J_\theta + J_\phi)^2}{4\pi^2} \cdot \frac{1}{r^2} \right]^{1/2} dr \end{aligned} \quad \dots (26)$$

Integrating by residue method* ; we get

$$J_r = -(J_\theta + J_\phi) + \pi k \sqrt{\left(\frac{2m}{-E} \right)} \quad \dots (27)$$

Thus we get

$$H = E = - \frac{2\pi^2 k^2 m}{(J_r + J_\theta + J_\phi)^2} \quad \dots (28)$$

Since J_r, J_θ, J_ϕ are symmetrical in the expression (28) ; therefore $\frac{\partial H}{\partial J_r}, \frac{\partial H}{\partial J_\theta}, \frac{\partial H}{\partial J_\phi}$ i.e. all the frequencies are equal and the motion is completely degenerate for a central force field of the form $V = -\frac{k}{r}$ and the required frequency of Kepler problem is given by

$$\nu = \frac{\partial H}{\partial J_r} = \frac{\partial H}{\partial J_\theta} = \frac{\partial H}{\partial J_\phi} = \frac{4\pi^2 k^2 m}{(J_r + J_\theta + J_\phi)^2} \quad \dots (29)$$

$$* \oint \left(A + \frac{2B}{r} - \frac{C}{r^2} \right) dr = 2\pi i \left\{ \sqrt{(-C)} + \frac{B}{\sqrt{A}} \right\} = 2\pi i \left\{ i\sqrt{C} + \frac{B}{\sqrt{A}} \right\}$$

15.14. Small Oscillations

Small oscillations means the oscillations about equilibrium position, when amplitude of motion is small such that for angular motion $\sin \theta = \theta$ and $\cos \theta = 1 - \frac{\theta^2}{2}$ may be approximated.

Under such a case the differential equation of simple harmonic motion is

$$\frac{d^2\theta}{dt^2} + \omega^2 \theta = 0 \quad \dots(1)$$

The solution of this equation is $\theta = \theta_0 e^{-i\omega t}$ where ω is angular frequency of motion. $\dots(2)$

In the case of simple pendulum $\omega = \sqrt{\frac{g}{l}}$ and time period $T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{l}{g}}$.

The simple mechanics of small oscillations, has been already discussed in separate chapter. Here we shall discuss the Lagrangian formulation of small oscillations. The theory of small oscillations finds its application in acoustics, molecular spectra, vibrations of mechanisms and coupled electrical circuits.

Stable and Unstable Equilibrium Positions. We consider conservative systems in which potential energy is function of position only. If $q_1, q_2, \dots, q_n$ are generalised coordinates of the system and transformation equations defining these generalised coordinates of the system do not involve time explicitly, then the system is said to be in equilibrium if the generalised forces acting on the system are zero, i.e.,

$$Q_k = - \left(\frac{\partial V}{\partial q_k} \right)_0 = 0$$

This implies potential energy of system in equilibrium is extremum. If system is in equilibrium initially when initial velocities $\dot{q}_k$ are zero, the system will remain in equilibrium indefinitely.

The examples of mechanical systems in equilibrium are : *a pendulum at rest, a suspension galvanometer at its zero position and an egg standing on an end.*

An equilibrium is said to be *stable* if small disturbance of the system from equilibrium results only in small bounded motion about the rest position. In terms of potential energy, the equilibrium is said to be stable if potential energy is *minimum* in rest position. The example of stable equilibrium is a pendulum at rest.

On the other hand an equilibrium position is said to be *unstable* if a small disturbance of the system from equilibrium position results an unbounded motion. In terms of potential energy, the equilibrium is said to be unstable if potential energy is *maximum* in equilibrium position. The example of unstable equilibrium is *an egg standing at one end*. Fig. 15-26(a) shows the position of stable equilibrium while Fig. 15-26(b) shows the position of unstable equilibrium.

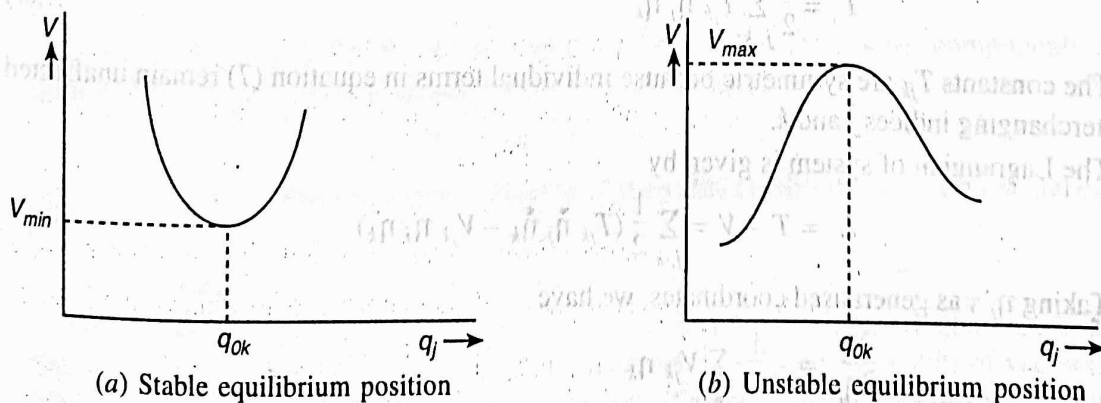


Fig. 15-26

We shall discuss the motion of system about the position of stable equilibrium when departures from equilibrium are small. Under this condition all functions may be expanded in a Taylor's Series about the equilibrium. If η_j is deviation of generalised coordinates q_{0j} then new generalised coordinate

$$q_j = q_{0j} + \eta_j \quad \dots(2)$$

The potential energy $V(q_1, q_2, \dots, q_n)$; (the function of new coordinates) may be expanded by Taylor's series about equilibrium function q_{0j} , i.e.,

$$V(q_1, q_2, \dots, q_n) = V(q_{01}, q_{02}, \dots, q_{0n}) + \sum_j \left(\frac{\partial V}{\partial q_j} \right)_0 \eta_j + \frac{1}{2} \sum_{j,k} \left(\frac{\partial^2 V}{\partial q_j \partial q_k} \right)_0 \eta_j \eta_k + \dots \dots(3)$$

In equilibrium position $\left(\frac{\partial V}{\partial q_k} \right)_0 = 0$ in view of equation (1), and first-term is potential energy in equilibrium position. Without loss of generality, it may be taken as zero by shifting the origin, so equation (3) becomes

$$V = \frac{1}{2} \sum_{j,k} \left(\frac{\partial^2 V}{\partial q_j \partial q_k} \right)_0 \eta_j \eta_k = \frac{1}{2} \sum_{j,k} V_{jk} \eta_j \eta_k \quad \dots(4)$$

where $V_{jk} = \left(\frac{\partial^2 V}{\partial q_j \partial q_k} \right)_0$ depend only on equilibrium values of q_j 's. By definition V_{jk} are symmetrical, i.e., $V_{jk} = V_{kj}$. The coefficients V_{jk} can be zero in several cases, so that potential energy can be independent of that particular coordinate. Consequently the equilibrium occurs at any arbitrary value of that coordinate. Such a case is said to be *neutral equilibrium* or *indifferent equilibrium*.

A similar series expansion can be carried out for kinetic energy. The kinetic energy is a homogenous quadratic function of generated velocities, i.e.,

$$\begin{aligned} T &= \sum_{j,k} \frac{1}{2} m_{jk} \dot{q}_j \dot{q}_k \text{ where } m_{kj} = m_{jk} \\ &= \sum_k \frac{1}{2} m_{jk} \dot{\eta}_j \dot{\eta}_k \quad \dots(5) \end{aligned}$$

The coefficients m_{jk} are general functions of coordinates q_j and may be expanded by Taylor series about equilibrium position.

$$m_{jk}(q_1, q_2, \dots, q_n) = m_{jk}(q_{01}, q_{02}, \dots, q_{0n}) + \sum_k \left(\frac{\partial m_{jk}}{\partial q_j} \right)_0 \eta_j + \dots \dots(6)$$

Equation (5) is already quadratic in $\dot{\eta}_j$'s, the lowest nonvanishing approximation to T is obtained by retaining only first term in the expansion of m_{jk} . Denoting constant values of m_{jk} at equilibrium by T_{jk} , we can write expression for kinetic energy as

$$T = \frac{1}{2} \sum_{j,k} T_{jk} \dot{\eta}_j \dot{\eta}_k \quad \dots(7)$$

The constants T_{jk} are symmetric because individual terms in equation (7) remain unaffected by interchanging indices j and k .

The Lagrangian of system is given by

$$L = T - V = \sum_{j,k} \frac{1}{2} (T_{jk} \dot{\eta}_j \dot{\eta}_k - V_{jk} \eta_j \eta_k)$$

Taking η_j 's as generalised coordinates, we have

$$\frac{\partial L}{\partial \eta_j} = -\frac{1}{2} \sum_k V_{jk} \eta_k$$

and
$$\frac{\partial L}{\partial \dot{\eta}_j} = \frac{1}{2} \sum_k T_{jk} \dot{\eta}_k$$

so the Lagrange's equation

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\eta}_j} \right) - \frac{\partial L}{\partial \eta_j} = 0 \text{ takes the form } \frac{d}{dt} \left(\sum_k \frac{1}{2} T_{jk} \dot{\eta}_k \right) + \sum_k \frac{1}{2} V_{jk} \eta_k = 0 \text{ with } j = 1, 2, \dots, n$$

$$\Rightarrow \sum_k (T_{jk} \ddot{\eta}_k + V_{jk} \eta_k) = 0 \quad \dots(8)$$

These are n -equations of motion. Each equation involves, in general, all the coordinates η_k . This set of simultaneous equations must be solved to describe the motions near equilibrium.

Eigenvalue Equation. Equation (8) represent linear differential equations with constant coefficients. Their oscillatory solutions may be expressed in the form

$$\eta_j = C a_j e^{-i\omega t} \quad \dots(9)$$

where $C a_j$ represent complex amplitude term for each coordinate; C is introduced as a scale factor merely for convenience. The actual motion is represented only by real part of equation (9). Substituting (8) into (7), we get for amplitude term

$$\sum_{jk} (V_{jk} a_k - \omega^2 T_{jk} a_k) = 0 \quad \dots(10)$$

This expression represents a set of n -linear homogeneous equations. Its has solution only if the determinant of coefficients is zero, i.e.,

$$\begin{vmatrix} V_{11} - \omega^2 T_{11} & V_{12} - \omega^2 T_{12} & \dots & V_{1n} - \omega^2 T_{1n} \\ V_{21} - \omega^2 T_{21} & V_{22} - \omega^2 T_{22} & \dots & V_{2n} - \omega^2 T_{2n} \\ V_{31} - \omega^2 T_{31} & V_{32} - \omega^2 T_{32} & \dots & V_{3n} - \omega^2 T_{3n} \\ \dots & \dots & \dots & \dots \\ V_{n1} - \omega^2 T_{n1} & V_{n2} - \omega^2 T_{n2} & \dots & V_{nn} - \omega^2 T_{nn} \end{vmatrix} = 0 \quad \dots(11)$$

This determinant represents an algebraic equation for ω^2 . The roots of determinant give the frequencies of motion considering η 's as generalised coordinates which are the Cartesian components multiplied by the square root of mass of particle (sometimes called mass-weighted coordinates), the kinetic energy can be put in the form (keeping in mind $T_{jk} = \delta_{jk}$)

$$T = \frac{1}{2} \sum_j \dot{\eta}_j^2 \quad \dots(12)$$

Also denoting ω^2 by λ , we get

$$\sum_k V_{jk} a_k = \lambda a_j \quad \dots(13)$$

Considering V_{jk} as the elements of an $n \times n$ matrix $\mathbf{V}$ and a_j as the components of an n dimensional column matrix $\mathbf{a}$, equation (13) may be expressed as

$$\mathbf{V}\mathbf{a} = \lambda\mathbf{a} \quad \dots(14)$$

This resembles with *eigenvalue equation*. Under this condition the determinental equation (11) reduces to secular equation for the eigen values λ .

As $\mathbf{U}$ is symmetrical and real, the corresponding eigen values are real.

If a matrix $\mathbf{A}$ is formed by n -sets of vectors $\mathbf{a}$'s corresponding to n -eigen values, then the matrix $\mathbf{A}$ must diagonalize $\mathbf{V}$ by similarity transformation. Moreover n -sets of vectors $\mathbf{a}$'s are orthogonal, so the diagonalising matrix $\mathbf{A}$ must be orthogonal. These conclusions hold whether

the matrix T_{ij} is diagonal or not. In view of this, equation (10) may represent an eigen value equation; considering T_{jk} as an element of matrix T , eigenvalue equations corresponding to equation (10) may be expressed as

$$\mathbf{V}\mathbf{a} = \lambda\mathbf{T}\mathbf{a} \quad \dots(15)$$

This equation represents an eigen value problem in which matrix operator $\mathbf{V}$ acting on column vector $\mathbf{a}$ produces a multiple of the result of operation of operator $\mathbf{T}$ on $\mathbf{a}$.

The matrices $\mathbf{T}$ and $\mathbf{V}$ are hermitian; the eigen values λ in equation (15) must be real and positive. This may be proved as follows :

Let $\mathbf{a}_j$ be the column matrix representing the j th eigen vector, satisfying the eigen value equation

$$\mathbf{V}\mathbf{a}_j = \lambda_j\mathbf{T}\mathbf{a}_j \quad \dots(16)$$

The transposed complex conjugate for λ_k has the form

$$\mathbf{a}_k^\dagger \mathbf{V} = \lambda_k^* \mathbf{a}_k^\dagger \mathbf{T} \quad \dots(17)$$

where $\mathbf{a}_k^\dagger$ is complex conjugate row matrix. It has been used that matrices $\mathbf{V}$ and $\mathbf{T}$ are hermitian (i.e. real and symmetric). Multiplying equation (17) from the right by $\mathbf{a}_j$, we get

$$\mathbf{a}_k^\dagger \mathbf{V} \mathbf{a}_j = \lambda_k^* \mathbf{a}_k^\dagger \mathbf{T} \mathbf{a}_j \quad \dots(18)$$

Multiplying equation (16) from left with $\mathbf{a}_k^\dagger$, we get

$$\mathbf{a}_k^\dagger \mathbf{V} \mathbf{a}_j = \lambda_j \mathbf{a}_k^\dagger \mathbf{T} \mathbf{a}_j \quad \dots(19)$$

Subtracting (17) from (18), we get

$$0 = (\lambda_j - \lambda_k^*) \mathbf{a}_k^\dagger \mathbf{T} \mathbf{a}_j \quad \dots(20)$$

For special case of $k = j$, we get

$$(\lambda_j - \lambda_j^*) \mathbf{a}_j^\dagger \mathbf{T} \mathbf{a}_j = 0$$

As $\mathbf{a}_j^\dagger \mathbf{T} \mathbf{a}_j \neq 0$, $\lambda_j = \lambda_j^*$, i.e., the eigen values λ_j must be real.

Multiplying equation (16) by $\tilde{\mathbf{a}}_j$ from the left (where $\tilde{\mathbf{a}}_j$ transpose of $\mathbf{a}_j$) we get

$$\tilde{\mathbf{a}}_j \mathbf{V} \mathbf{a}_j = \lambda_j \tilde{\mathbf{a}}_j \mathbf{T} \mathbf{a}_j$$

we get

$$\lambda_j = \frac{\tilde{\mathbf{a}}_j \mathbf{V} \mathbf{a}_j}{\tilde{\mathbf{a}}_j \mathbf{T} \mathbf{a}_j} \quad \dots(21)$$

The numerator of this equation is positive for coordinates a_{jk} due to minimum value of potential energy in equilibrium and the denominator is also positive because it represents twice the kinetic energy for velocities a_{jk} . Hence eigen values λ_j are definitely *real* and *positive*.

In view of this equation (20) becomes

$$(\lambda_j - \lambda_k) \tilde{\mathbf{a}}_k \mathbf{T} \mathbf{a}_j = 0 \quad \dots(22a)$$

If $\lambda_j \neq \lambda_k$, then $\tilde{\mathbf{a}}_k \mathbf{T} \mathbf{a}_j = 0$

In case $\lambda_j = \lambda_k$, then $\tilde{\mathbf{a}}_k \mathbf{T} \mathbf{a}_j$ may be taken as identity matrix, i.e.,

$$\tilde{\mathbf{a}}_j \mathbf{T} \mathbf{a}_j = 1 \quad \dots(22b)$$

If two or more eigen values are equal, the system is said to be *degenerate*, however for the present we shall consider them to be distinct.

$$\text{As } \lambda = \omega^2 \Rightarrow \omega = \sqrt{\lambda}$$

The general solutions of the equation of motions may be written as

$$\eta_j = C_k a_{jk} e^{-i\omega_k t} \quad \dots(23a)$$

C_k is complex scale factor for each resonant frequency. As ω_k are real, the actual solution will be represented by real part of (23a), i.e.,

$$\eta_j = f_k a_{jk} \cos(\omega_k t + \delta) \quad \dots(23b)$$

where amplitude f_k and phase δ_k are determined from the initial condition. Equations (23) represents actual solution.

The solution for each coordinate equation (23) is in general a sum of simple harmonic oscillations in all of the frequencies ω_k , satisfying the similar equation. It is possible to transform from coordinates η_j to a new set of generalised coordinates that are all simple periodic functions of time. This new set of variable are called the *normal coordinates*. The transformation equations relating new set of coordinates ξ_k to original coordinates η_j are

$$\eta_j = \sum_k a_{jk} \xi_k \quad \dots(24)$$

In terms of single column matrices η and ξ

$$\eta = A \xi$$

The potential energy in terms of new coordinates is

$$V = \sum_k \frac{1}{2} \lambda_k \xi_k^2 = \sum_k \frac{1}{2} \omega_k^2 \xi_k^2$$

$$\text{and Kinetic energy } T = \sum_k \frac{1}{2} \dot{\xi}_k \dot{\xi}_k$$

Lagrangian $L = T - V$

$$= \frac{1}{2} \sum_k \left(\dot{\xi}_k^2 - \omega_k^2 \xi_k^2 \right) \quad \dots(25)$$

Lagrange's equation for ξ_k 's are $\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\xi}_k} \right) - \frac{\partial L}{\partial \xi_k} = 0$

$$\text{From (25)} \quad \frac{\partial L}{\partial \dot{\xi}_k} = \dot{\xi}_k \quad \text{and} \quad \frac{\partial L}{\partial \xi_k} = -\omega_k^2 \xi_k$$

Lagrange's equation becomes

$$\frac{d}{dt} (\dot{\xi}_k) + \omega_k^2 \xi_k = 0$$

$$\Rightarrow \ddot{\xi}_k + \omega_k^2 \xi_k = 0$$

Its solution is

$$\xi_k = C_k e^{-i\omega_k t} \quad \dots(26)$$

This is simply periodic in ξ_k involving only one of the resonant frequencies. This justifies the name ξ_k 's as normal coordinates of the system. Each normal coordinate corresponds to vibration of system with only one frequency ω_k ; these component vibrations are said to be *normal modes of vibrations*. All the particles in one mode vibrate with the *same frequency and with the same phase*.

The complete motion consists of the superposition of normal modes weighed with appropriate amplitude and phase factors contained in the constants C_k 's. It may be noted that the Lagrangian of the system is the sum of the Lagrangian for harmonic oscillators of frequencies ω_k .

15.14 (a). Example of the Double Pendulum as a Coupled Oscillator

Suppose there are two pendulums of equal mass and length suspended from a non-rigid support so that the displacement of one pendulum affects the potential energy of the other and *vice-versa*.

Let us first consider the motion of a simple pendulum of mass m and length l inclined at θ_1 to the vertical. At this instant let s_1 be the arc length along the circular path. Then

$$s_1 = l\theta_1 \quad \dots(1)$$

Let T_1 and V_1 be its kinetic and potential energies respectively.

$$T_1 = \frac{1}{2} m \dot{s}_1^2 = \frac{1}{2} m l^2 \dot{\theta}_1^2, \quad \dots(2)$$

since from (1), $\dot{s}_1 = l\dot{\theta}_1$ and $V_1 = mg[l - l \cos \theta_1]$

$$= mgl(1 - \cos \theta_1). \quad \dots(3)$$

Using the natural units such that m, l, g all can be made equal to unity, i.e., $m = l = g = 1$, then equations (2) and (3) give

$$T_1 = \frac{1}{2} \dot{\theta}_1^2, \quad \dots(4)$$

$$\text{and } V_1 = 1 - \cos \theta_1 = 2 \sin^2 \frac{\theta_1}{2} = 2 \left[\frac{\theta_1}{2} - \frac{1}{3!} \left(\frac{\theta_1}{2} \right)^3 + \dots \right]^2$$

But we are considering small oscillations, therefore higher powers of θ_1 can be neglected.

We thus have

$$V_1 = 2 \left(\frac{\theta_1}{2} \right)^2 = \frac{1}{2} \theta_1^2. \quad \dots(5)$$

Let us now assume that T is the kinetic energy and V the potential energy of the coupled oscillator, i.e., double pendulum, then the results (4) and (5) derived for a single pendulum will appear in the following form for a double pendulum,

$$T = \frac{1}{2} (\dot{\theta}_1^2 + \dot{\theta}_2^2), \quad \dots(6)$$

$$V = \frac{1}{2} (\theta_1^2 + \theta_2^2 - 2\kappa \theta_1 \theta_2), \quad \dots(7)$$

where κ is a coupling constant dependent on the actual structure of the pendulum support and θ_2 is the angle of the second pendulum from the vertical.

The two symmetric matrices T_{jk} and V_{jk} defined in 15.14, may be expressed here as

$$T_{jk} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } V_{jk} = \begin{bmatrix} 1 & -\kappa \\ -\kappa & 1 \end{bmatrix} \quad \dots(8)$$

and the eigen frequencies will be given by

$$|V_{jk} - \omega^2 T_{jk}| = 0, \text{ i.e., } \begin{vmatrix} 1 - \omega^2 & -\kappa \\ -\kappa & 1 - \omega^2 \end{vmatrix} = 0 \quad \dots(9)$$

Its solution gives

$$(1 - \omega^2)^2 - \kappa^2 = 0$$

or

$$1 - \omega^2 = \pm \kappa \quad \dots(10)$$

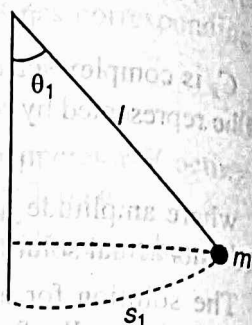


Fig. 15.27

If κ is very small we have

$$\omega = (1 \mp \kappa)^{1/2} \quad \text{... (11)}$$

i.e.,

$$\omega = 1 \mp \frac{1}{2} \kappa$$

expanding binomially and neglecting higher order terms.

The result (11) in conventional units becomes

$$\omega = \sqrt{\left(\frac{g}{l}\right)} \left(1 \mp \frac{1}{2} \kappa\right) \quad \text{... (12)}$$

Using (16) of 15.14, the normal modes of oscillations are given by

$$\left. \begin{aligned} \pm \kappa \theta_1 - \kappa \theta_2 &= 0 \\ -\kappa \theta_1 \pm \kappa \theta_2 &= 0 \end{aligned} \right\} \quad \text{... (13)}$$

either of which may yield

$$\frac{\theta_1}{\theta_2} = \pm 1.$$

Case I. If $\frac{\theta_1}{\theta_2} = +1$ and $\kappa > 0$, the pendulums oscillate together with an angular frequency less than that of one of the pendulums simply.

Case II. If $\frac{\theta_1}{\theta_2} = -1$ and $\kappa > 0$, the two pendulums oscillate in opposite directions at a frequency larger than the uncoupled frequency.

In general the combined oscillation will be the superposition of the two modes of oscillations and the phenomenon of beats is shown by the oscillations of either pendulum.

15.14 (b). Example of Triple Pendulum as a Degenerate System

Suppose there are three pendulums of equal mass and joined to each other symmetrically such that the third pendulum is suspended symmetrically between the two pendulums and the coupling exists among the pendulums through the same flexible support.

Using the natural units such that m (mass), l (length) and g all can be made equal to unity i.e., $m = l = g = 1$, and borrowing notations from 15.14, the equations (6) and 7 of 15.14 giving the kinetic energy (T) and potential energy (V) of the triple pendulum now take the form

$$T = \frac{1}{2} (\dot{\theta}_1^2 + \dot{\theta}_2^2 + \dot{\theta}_3^2) \quad \text{... (1)}$$

$$V = \frac{1}{2} [\theta_1^2 + \theta_2^2 + \theta_3^2 - 2\kappa (\theta_1 \theta_2 + \theta_2 \theta_3 + \theta_3 \theta_1)]. \quad \text{... (2)}$$

where κ is a coupling constant depending on the actual structure of the pendulum and θ_i ($i = 1, 2, 3$) is the angle which the i th pendulum makes with the vertical.

In this case the two symmetric matrices T_{kj} and V_{kj} defined in 15.14 may be expressed as

$$T_{jk} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad V_{jk} = \begin{bmatrix} 1 & -\kappa & -\kappa \\ -\kappa & 1 & -\kappa \\ -\kappa & -\kappa & 1 \end{bmatrix} \quad \text{... (3)}$$

and the eigen frequencies are given by

$$|V_{kj} - \omega^2 T_{kj}| = 0$$

i.e.,

$$\begin{vmatrix} 1 - \omega^2 & -\kappa & -\kappa \\ -\kappa & 1 - \omega^2 & -\kappa \\ -\kappa & -\kappa & 1 - \omega^2 \end{vmatrix} = 0 \quad \text{... (4)}$$

which gives on expansion, $(1 - \omega^2)^3 - 3\kappa^2(1 - \omega^2) - 2\kappa^2 = 0$

$$\text{or} \quad (1 - \omega^2 + \kappa)^2 (1 - \omega^2 - 2\kappa) = 0$$

$$\text{giving} \quad 1 - \omega^2 = -\kappa, -\kappa, 2\kappa$$

$$\text{i.e.,} \quad \omega^2 = 1 + \kappa, 1 + \kappa, 1 - 2\kappa.$$

Two frequencies being identical, the system is clearly **degenerate**.

If κ is very small, we have

$$\text{and} \quad \left. \begin{aligned} \omega &= (1 + \kappa)^{1/2} = 1 + \frac{\kappa}{2} \\ \omega &= (1 - 2\kappa)^{1/2} = 1 - \kappa \end{aligned} \right\}$$

expanding binomially and neglecting higher order terms.

The result (6) in conventional units becomes

$$\omega = \sqrt{\left(\frac{g}{l}\right) \left(1 + \frac{1}{2}\kappa\right)} + \sqrt{\left(\frac{g}{l}\right) \left(1 + \frac{1}{2}\kappa\right)} + \sqrt{\left(\frac{g}{l}\right) (1 - \kappa)}$$

Now using (16) of 15.14 the normal modes of oscillation can be found as follows.

Eigen vectors are given by

$$(V - T\omega^2) \xi = 0,$$

$$\text{i.e.,} \quad \begin{bmatrix} 1 - \omega^2 & -\kappa & -\kappa \\ -\kappa & 1 - \omega^2 & -\kappa \\ -\kappa & -\kappa & 1 - \omega^2 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = 0$$

$$\text{or equivalently,} \quad \left. \begin{aligned} (1 - \omega^2) \xi_1 - \kappa \xi_2 - \kappa \xi_3 &= 0 \\ -\kappa \xi_1 + (1 - \omega^2) \xi_2 - \kappa \xi_3 &= 0 \\ -\kappa \xi_1 - \kappa \xi_2 + (1 - \omega^2) \xi_3 &= 0 \end{aligned} \right\}$$

Let $\xi_{11}, \xi_{12}, \xi_{13}$ be the components of ξ_1 , corresponding to $1 - \omega^2 = -\kappa$. Then (3) give

$$\xi_{11} + \xi_{12} + \xi_{13} = 0$$

Choosing $\xi_{12} = 0, \xi_{11} = -\xi_{13} = \alpha$ (say).

$$\text{So} \quad \xi_1 = \alpha (1, 0, -1) = \frac{1}{\sqrt{2}} (1, 0, -1)$$

taking $\alpha = \frac{1}{\sqrt{2}}$ subject to normalization condition i.e., $\alpha \sqrt{1^2 + 0^2 + (-1)^2} = 1$.

Also if $\xi_{21}, \xi_{22}, \xi_{23}$ be the components of ξ_2 , corresponding to another frequency $1 - \omega^2 = -\kappa$, then (8) give

$$\xi_{21} + \xi_{22} + \xi_{23} = 0. \quad (10)$$

Now the second eigen vector ξ_2 cannot be determined by this equation and so if we choose the second eigen vector ξ_2 such that it is orthogonal to ξ_1 , then orthogonality condition for two vectors gives

$$\xi_1 \cdot \xi_2 = 0$$

$$\text{i.e.,} \quad [1, 0, -1] \begin{bmatrix} \xi_{21} \\ \xi_{22} \\ \xi_{23} \end{bmatrix} = 0$$

$$\text{or} \quad \text{equivalently } \xi_{21} - \xi_{23} = 0.$$

(10) and (11) gives $\xi_{21} = \xi_{23} = \beta$ (say) and $\xi_{22} = -2\beta$.

So $\xi_2 = \beta (1, -2, -1) = \frac{1}{\sqrt{6}} (1, -2, 1) \dots (12)$

subject of normalization condition.

$$\sqrt{\xi_{21}^2 + \xi_{22}^2 + \xi_{23}^2} = 1.$$

Finally if $\xi_{31}, \xi_{32}, \xi_{33}$ be the components of ξ_3 , corresponding to $1 - \omega^2 = 2\kappa$, then (8) gives

$$\begin{aligned} 2\xi_{31} - \xi_{32} - \xi_{33} &= 0 \\ -\xi_{31} + \xi_{32} + 2\xi_{33} &= 0 \\ -\xi_{31} - \xi_{32} + 2\xi_{33} &= 0 \end{aligned}$$

which yield, $\xi_{31} = \xi_{32} = \xi_{33} = \gamma$ (say).

$\therefore \xi_3 = \gamma (1, 1, 1) = \frac{1}{\sqrt{3}} (1, 1, 1) \dots (13)$

$$\sqrt{(\gamma^2 + \gamma^2 + \gamma^2)} = 1 \text{ for normalization.}$$

By (19) of 15.14, the normal coordinates are given by

$$x_k = C_k e^{-i\omega_k t}, k = 1, 2, 3 \text{ and correspondingly}$$

$$\omega_1 = \omega_2 = \sqrt{\left(\frac{g}{l}\right) \left(1 + \frac{1}{2}\kappa\right)}$$

$$\omega_3 = \sqrt{\left(\frac{g}{l}\right) (1 - \kappa)}.$$

In other words, since normal coordinates x_k are along the directions of the eigen vector ξ_k ($k = 1, 2, 3$), so by (9), (12) and (13) we have

$$x_1 = \frac{1}{\sqrt{2}} (\theta_1 - \theta_3),$$

$$x_2 = \frac{1}{\sqrt{6}} (\theta_1 - 2\theta_2 + \theta_3),$$

$$x_3 = \frac{1}{\sqrt{3}} (\theta_1 + \theta_2 + \theta_3).$$

15.14 (c). Free Vibrations of a Linear Triatomic Molecule

Consider the equilibrium configuration of the molecule such that two of its atoms of each of mass m are symmetrically placed on each side of the third atom of mass M . All three atoms are collinear. Assuming the motion along the line of molecules and there being no interaction between the end atoms, let κ be the force constant approximated by two strings of force joining the three atoms.

Taking η_1, η_2, η_3 the displacements of each atom from the equilibrium configuration, the Kinetic and Potential energies are given by

$$T = \frac{1}{2} m (\dot{\eta}_1^2 + \dot{\eta}_3^2) + \frac{1}{2} M \dot{\eta}_2^2 \dots (1)$$

$$\begin{aligned} V &= \frac{1}{2} \kappa (\eta_2 - \eta_1)^2 + \frac{1}{2} \kappa (\eta_3 - \eta_2)^2 \\ &= \frac{\kappa}{2} (\eta_1^2 + 2\eta_2^2 + \eta_3^2 - 2\eta_1 \eta_2 - 2\eta_2 \eta_3). \dots (2) \end{aligned}$$

In this case the two symmetric matrices T_{jk} and V_{jk} are

$$T_{jk} = \begin{bmatrix} m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & m \end{bmatrix} \text{ and } V_{jk} = \begin{bmatrix} \kappa & \kappa & 0 \\ -\kappa & 2\kappa & -\kappa \\ 0 & -\kappa & \kappa \end{bmatrix} \quad \dots(3)$$

and the eigen frequencies are given by

$$|V_{kj} - \omega^2 T_{kj}| = 0$$

$$\text{i.e., } \begin{vmatrix} \kappa - \omega^2 m & -\kappa & 0 \\ -\kappa & 2\kappa - \omega^2 M & -\kappa \\ 0 & -\kappa & \kappa - \omega^2 m \end{vmatrix} = 0 \quad \dots(4)$$

which gives on expansion

$$\omega^2 \kappa - \omega^2 m [\kappa (M + 2m) - \omega^2 m M] = 0$$

$$\text{i.e., } \omega^2 = 0, \frac{\kappa}{m}, \frac{\kappa (M + 2m)}{mM}$$

so that three normal frequencies are

$$\omega_1 = 0, \omega_2 = \sqrt{\left(\frac{\kappa}{m}\right)}, \omega_3 = \sqrt{\left(\frac{\kappa}{m} \left\{1 + \frac{2m}{M}\right\}\right)}, \quad \dots(5)$$

Here the first eigen value $\omega_1 = 0$ does not correspond to an oscillatory motion, since the equation of motion for the corresponding normal coordinate is $\ddot{x}_k = 0$ {by (18) of 15.14} which actually produces a uniform translatory motion. Such a vanishing frequency is caused by the molecule translated rigidly along its axis without any change in the potential energy, since restoring force against such motion is zero.

Let ξ_1, ξ_2, ξ_3 be the given vectors corresponding to $\omega_1, \omega_2, \omega_3$ respectively and let components of ξ_1 be $\xi_{11}, \xi_{12}, \xi_{13}$, those of ξ_2 be $\xi_{21}, \xi_{22}, \xi_{23}$ and those of ξ_3 be $\xi_{31}, \xi_{32}, \xi_{33}$.

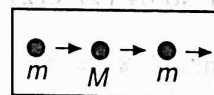


Fig. 15.28(a)

$$\begin{bmatrix} \kappa & -\kappa & 0 \\ -\kappa & 2\kappa & -\kappa \\ 0 & -\kappa & \kappa \end{bmatrix} \begin{bmatrix} \xi_{11} \\ \xi_{12} \\ \xi_{13} \end{bmatrix} = 0$$

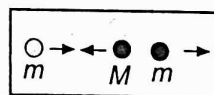


Fig. 15.28(b)

or equivalently $\xi_{11} - \xi_{12} = 0, -\xi_{11} + \xi_{12} - \xi_{13} = 0, -\xi_{12} + \xi_{13} = 0$

which yield $\xi_{11} = \xi_{12} = \xi_{13} = \alpha$ (say),

so $\xi_1 = \alpha(1, 1, 1) = \begin{bmatrix} \alpha \\ \alpha \\ \alpha \end{bmatrix}$

which follows that each atom is equally displaced.

Now $\omega_2 = \sqrt{\left(\frac{\kappa}{m}\right)}$ gives

$$\begin{bmatrix} 0 & -\kappa & 0 \\ -\kappa & 2\kappa - \kappa (M/m) & -\kappa \\ 0 & -\kappa & 0 \end{bmatrix} \begin{bmatrix} \xi_{21} \\ \xi_{22} \\ \xi_{23} \end{bmatrix} = 0$$

or equivalently $\xi_{22} = 0$, $\xi_{21} + \xi_{23} = 0$, i.e., $\xi_{21} = -\xi_{23} = \beta$ (say),

So $2\xi_2 = \beta (1, 0, -1) = \begin{bmatrix} \beta \\ 0 \\ -\beta \end{bmatrix} \quad \dots(7)$

which follows that the central atom remains at rest while the end atoms are displaced equally in opposite sense.

Again $\omega_3 = \sqrt{\left\{ \frac{\kappa}{m} \left(1 + \frac{2m}{M} \right) \right\}}$ gives

$$\begin{bmatrix} -(2m/M) & \kappa & 0 \\ -\kappa & -(M/m) & \kappa \\ 0 & -\kappa & -(2m/M) \end{bmatrix} \begin{bmatrix} \xi_{31} \\ \xi_{32} \\ \xi_{33} \end{bmatrix} = 0$$

or equivalently

$$\frac{2m}{M} \xi_{31} + \kappa \xi_{32} = 0$$

$$\kappa \xi_{31} + \kappa \frac{M}{m} \xi_{32} + \kappa \xi_{33} = 0,$$

$$\kappa \xi_{32} + \frac{2m}{M} \xi_{33} = 0,$$

which yields $\xi_{31} = \xi_{33} = \gamma$ (say) and $\xi_{32} = \frac{2m}{M} \xi_{33} = -\frac{2m}{M} \gamma$.

$$\therefore \xi^2 = \gamma [1, -(2m/M), 1] = \begin{bmatrix} \gamma \\ (-2m/(M) \gamma) \\ \gamma \end{bmatrix} \quad \dots(8)$$

which follows that the displacement of end atoms is equal while that of the central atom is equal to neither of end one's and moreover differs in phase.

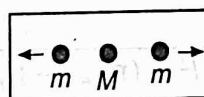


Fig. 15-28(c)

Now ξ will be orthogonal, if

$$\xi^T \bar{\xi} = I, \quad \dots(9)$$

where I is the unit matrix.

Here $\xi = \begin{bmatrix} \xi_{11} & \xi_{12} & \xi_{13} \\ \xi_{21} & \xi_{22} & \xi_{23} \\ \xi_{31} & \xi_{32} & \xi_{33} \end{bmatrix} = \begin{bmatrix} \alpha & \alpha & \alpha \\ \beta & 0 & -\beta \\ \gamma & -(2m/M)\gamma & \gamma \end{bmatrix}$

and $\bar{\xi}$ (or ξ^T) is its transpose

Also $T = \begin{bmatrix} m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & m \end{bmatrix}$

$\therefore$ Equation (9) gives

$$\begin{bmatrix} \alpha & \alpha & \alpha \\ \beta & 0 & -\beta \\ \gamma & -(2m/M)\gamma & \gamma \end{bmatrix} \begin{bmatrix} m & 0 & 0 \\ 0 & M & 0 \\ \gamma & 0 & m \end{bmatrix} \begin{bmatrix} \alpha & \beta & \gamma \\ \alpha & 0 & -(2m/M)\gamma \\ \alpha & -\beta & \gamma \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{or } \begin{bmatrix} \alpha & \alpha & \alpha \\ \beta & 0 & -\beta \\ \gamma & -(2m/M) & \gamma \end{bmatrix} \begin{bmatrix} m\alpha & m\beta & m\gamma \\ M\beta & 0 & -2m\gamma \\ m\alpha & m\beta & m\gamma \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{or } \begin{bmatrix} (2m + M)\alpha^2 & 0 & 0 \\ 0 & 2m\beta^2 & 0 \\ 0 & 0 & 2m\gamma^2 + (4m^2/M)\gamma^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{or equivalently } (2m + M)\alpha^2 = 1, \text{ i.e., } \alpha = \frac{1}{\sqrt{(2m + M)}}$$

$$2m\beta^2 = 1, \text{ i.e., } \beta = \frac{1}{\sqrt{(2m)}}$$

$$\left(2m + \frac{4m^2}{M}\right)\gamma^2 = 1, \text{ i.e., } \gamma = \frac{1}{\sqrt{\left\{2m \left(1 + \frac{2m}{M}\right)\right\}}}$$

15.14. (d) General Case, i.e., System with n Degrees of Freedom

Consider a light elastic string of length $(n + 1)l$ loaded with n particles, each of mass m and placed at distances $l, 2l, 3l, \dots, nl$ from one end A (say). Suppose that two ends A and B of the string have been fixed and gravity does not come into action, also that the tension in the string provides the restoring force. Let the displacements of the particles vibrating in a direction perpendicular to the string be respectively $\eta_1, \eta_2, \eta_3, \dots, \eta_n$. Then kinetic energy of the system is given by

$$T = \frac{1}{2} m (\dot{\eta}_1^2 + \dot{\eta}_2^2 + \dots + \dot{\eta}_n^2). \quad \dots(1)$$

To find the potential energy, consider the distance between j th and $(j + 1)$ th particles which is

$$\begin{aligned} x_{j,j+1} &= \sqrt{\{l^2 + (\eta_{j+1} - \eta_j)^2\}} \\ &= l \left\{ 1 + \frac{1}{l^2} (\eta_{j+1} - \eta_j)^2 \right\}^{1/2} \\ &= l \left\{ 1 + \frac{1}{2} \cdot \frac{1}{l^2} (\eta_{j+1} - \eta_j)^2 \right\} \end{aligned}$$

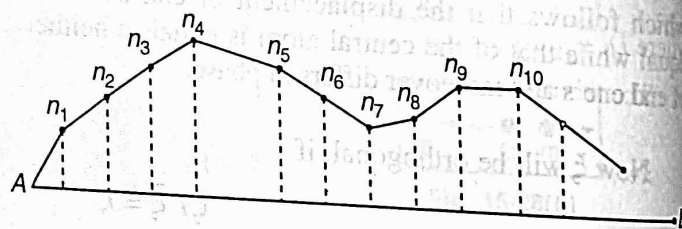


Fig. 15-29

expanding binomially and retaining only first terms for small displacement.

$$x_{j,j+1} = l + \frac{1}{2l} (\eta_{j+1} - \eta_j)^2.$$

$\therefore$ Extension of this portion of the string

$$\begin{aligned} &= l + \frac{1}{2l} (\eta_{j+1} - \eta_j)^2 - l. \\ &= \frac{1}{2l} (\eta_{j+1} - \eta_j)^2 \end{aligned}$$

Taking κ as the tension of the string, the potential energy being 'tension times length' is given for this segment by

$$\frac{\kappa}{2l} (\eta_{j+1} - \eta_j)^2.$$

Hence the total potential energy of the system is given by

$$\begin{aligned} V &= \frac{\kappa}{2l} \sum_{j=0}^n (\eta_{j+1} - \eta_j)^2 \\ &= \frac{\kappa}{2l} [(\eta_1)^2 + (\eta_2 - \eta_1)^2 + (\eta_3 - \eta_2)^2 + \dots + (\eta_n - \eta_{n-1})^2 + (\eta_{n+1} - \eta_n)^2] \\ &= \frac{\kappa}{2l} [2\eta_1^2 + 2\eta_2^2 + \dots + 2\eta_n^2 - 2\eta_1\eta_2 - 2\eta_2\eta_3 - \dots - 2\eta_{n-1}\eta_n] \quad \dots(2) \end{aligned}$$

under the assumption that $\eta_0 = \eta_{n+1} = 0$

Choosing the units such that $m = 1$, $\kappa/l = 1$, (1) and (2) become

$$T = \frac{1}{2} (\dot{\eta}_1^2 + \dot{\eta}_2^2 + \dots + \dot{\eta}_n^2) \quad \dots(3)$$

$$V = \frac{1}{2} (2\eta_1^2 + 2\eta_2^2 + \dots + 2\eta_n^2 - 2\eta_1\eta_2 - 2\eta_2\eta_3 - \dots - 2\eta_{n-1}\eta_n) \quad \dots(4)$$

Thus the symmetric matrices T_{kj} and V_{kj} are

$$T_{kj} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{bmatrix} \text{ and } V_{kj} = \begin{bmatrix} 2 & -1 & 0 & \dots & 0 \\ 1 & 2 & -1 & \dots & 0 \\ 0 & -1 & 2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 2 \end{bmatrix} \quad \dots(5)$$

So that $|V_{kj} - \omega^2 T_{kj}| = 0$ gives

$$(\text{say}) \quad D_n = \begin{bmatrix} 2 - \omega^2 & -1 & 0 & \dots & 0 \\ -1 & 2 - \omega^2 & -1 & \dots & 0 \\ 0 & -1 & 2 - \omega^2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 2 - \omega^2 \end{bmatrix} = 0, \quad \dots(6)$$

where D_n is the determinant of order n .

The theory of determinant yields

$$\text{i.e., } \left. \begin{aligned} D_n &= (2 - \omega^2) D_{n-1} - D_{n-2} \\ D_n - (2 - \omega^2) D_{n-1} + D_{n-2} &= 0 \end{aligned} \right\} \quad \dots(7)$$

$$\text{Putting } 2 - \omega^2 = 2 \cos \theta, \text{ i.e., } \omega^2 = 4 \sin^2 \frac{\theta}{2}, \quad \dots(8)$$

$$\text{we have } D_n - 2 \cos \theta D_{n-1} + D_{n-2} = 0. \quad \dots(9)$$

Assuming $D_n = Z^n$ as a trial solution for this equation, we have

$$Z^2 - 2 \cos \theta Z + 1 = 0, \quad \dots(10)$$

giving

$$Z = e^{i\theta}, e^{-i\theta} \quad \dots(10)$$

As such the solution of equation (9) is

$$D_n = A e^{in\theta} + B e^{-in\theta}, \quad \dots(11)$$

where A, B are arbitrary functions of θ .

From equation (11), we find

$$\left. \begin{aligned} D_1 &= A e^{i\theta} + B e^{-i\theta} \\ D_2 &= A e^{2i\theta} + B e^{-2i\theta} \end{aligned} \right\} \quad \dots(12)$$

while from equation (6),

$$\left. \begin{aligned} D_1 &= (2 - \omega^2) = 2 \cos \theta \\ D_2 &= (2 - \omega^2)^2 - 1 = (2 \cos \theta)^2 - 1 = 4 \cos^2 \theta - 1 \end{aligned} \right\} \dots(13)$$

Evidently the equations (12) and (13) are simultaneously satisfied if

$$A = \frac{e^{i\theta}}{2i \sin \theta} \text{ and } B = \frac{e^{-i\theta}}{2i \sin \theta}$$

Substituting these values of A and B in (11), we get

$$D_n = \frac{e^{i(n+1)\theta}}{2i \sin \theta} - \frac{e^{-i(n+1)\theta}}{2i \sin \theta} \dots(14)$$

Substituting these values of A and B in (11), we get

$$\begin{aligned} D_n &= \frac{e^{i(n+1)\theta}}{2i \sin \theta} - \frac{e^{-i(n+1)\theta}}{2i \sin \theta} \\ &= \frac{2i \sin (n+1)\theta}{2i \sin \theta} \\ &= \frac{\sin (n+1)\theta}{\sin \theta} \end{aligned} \dots(15)$$

Thus roots of (6) i.e., $D_n = 0$ are given by

$$\frac{\sin (n+1)\theta}{\sin \theta} = 0$$

or

$$\sin (n+1)\theta = 0$$

i.e.,

$$\theta_r = \frac{r\pi}{n+1}, r = 1, 2, \dots, n. \dots(16)$$

and the corresponding eigen values are given by (8) as

$$\begin{aligned} \omega_2^2 &= 4 \sin^2 \frac{\theta_r}{2} = 4 \sin^2 \frac{r\pi}{2(n+1)} \\ r &= 1, 2, \dots, n. \end{aligned} \dots(17)$$

Thus normal frequencies etc. can be determined.

SOLVED EXAMPLES

Ex. 52. A particle of mass m can move along a straight line and is attached to a spring whose other end is fixed to a point O . The length of spring in equilibrium is l . When restoring force is F , the elongation of the spring is δl . Find the frequency of small oscillations of mass m .

Solution. Potential energy U of spring
= force $\times$ extension = $F \delta l$

From Fig. 15.30 $(l + \delta l)^2 = l^2 + x^2$

$$\Rightarrow l^2 + \delta l^2 + 2l \delta l = l^2 + x^2$$

As δl is small, δl^2 can be neglected, so above equation gives

$$\delta l \approx \frac{x^2}{2l}$$

$$\therefore \text{Potential energy of spring } U = F \cdot \frac{x^2}{2l} = \frac{1}{2} \frac{Fx^2}{l}$$

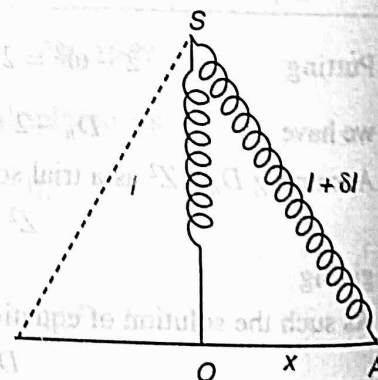


Fig. 15.30

Kinetic energy of mass $T = \frac{1}{2} m \dot{x}^2$

$$\therefore \text{Lagrangian of system } L = T - V = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} \frac{Fx^2}{l} \quad \dots(1)$$

Lagrange's equation is

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = 0 \quad \dots(2)$$

$$\text{From (1) } \frac{\partial L}{\partial \dot{x}} = m \dot{x}, \quad \frac{\partial L}{\partial x} = \frac{F}{l} x$$

Equation (2) gives

$$m \ddot{x} - \frac{F}{l} x = 0$$

$$\Rightarrow \ddot{x} - \frac{F}{ml} x = 0$$

Standard equation of SHM is $\ddot{x} - \omega^2 x = 0$

$$\therefore \text{Frequency of motion, } f = \frac{\omega}{2\pi} = \frac{1}{2\pi} \sqrt{\frac{F}{ml}}$$

Ex. 53. Find the frequency of small oscillations of a simple pendulum whose point of support can move horizontally.

Solution. The system has two degrees of freedom. The generalised coordinates are x and θ (Fig. 15-31)

If m_1 is mass of support and m_2 that of bob, then

$$\text{mass } m_1, \text{ kinetic energy } T_1 = \frac{1}{2} m_1 \dot{x}^2$$

Potential energy, $V_1 = \text{zero}$

$$\text{For mass } m_2, \text{ kinetic energy, } T_2 = \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2)$$

$$= \frac{1}{2} m_2 \left[\left\{ \frac{d}{dt} (x + l \sin \theta) \right\}^2 + \left\{ \frac{d}{dt} (l \cos \theta) \right\}^2 \right] \quad \text{Fig. 15-31}$$

$$= \frac{1}{2} m_2 [(\dot{x} + l \cos \theta \dot{\theta})^2 + (-l \sin \theta \dot{\theta})^2]$$

$$= \frac{1}{2} m_2 [\dot{x}^2 + l^2 \dot{\theta}^2 + 2\dot{x}\dot{\theta} l \cos \theta]$$

$$\text{Potential energy } V_2 = -m_2 g l \cos \theta$$

$$\text{Lagrangian of system } L = T - V = (T_1 + T_2) - (V_1 + V_2)$$

$$\Rightarrow L = \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 (\dot{x}^2 + l^2 \dot{\theta}^2 + 2\dot{x}\dot{\theta} l \cos \theta) + m_2 g l \cos \theta \quad \dots(1)$$

$$\frac{\partial L}{\partial x} = 0, \quad \frac{\partial L}{\partial \dot{x}} = m_1 \dot{x} + m_2 \dot{x} + m_2 \dot{\theta} l \cos \theta$$

$$\frac{\partial L}{\partial \theta} = m_2 l^2 \ddot{\theta} + m_2 \dot{x} l \cos \theta, \quad \frac{\partial L}{\partial \dot{\theta}} = -m_2 l \dot{x} \sin \theta - m_2 g l \sin \theta$$

Lagrangian equation in coordinate x is

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = 0$$

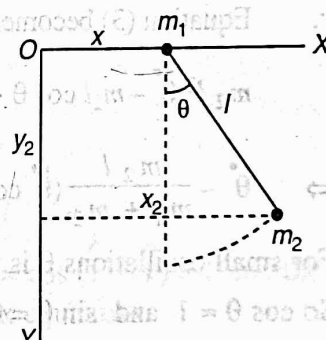


Fig. 15-31

$$\Rightarrow \frac{d}{dt} \{ (m_1 + m_2) \dot{x} + m_2 l \dot{\theta} \cos \theta \} = 0$$

$$\Rightarrow (m_1 + m_2) \ddot{x} + m_2 l \cos \theta \ddot{\theta} - m_2 l \dot{\theta}^2 \sin \theta = 0 \quad \dots(2)$$

Lagrangian equation in coordinate θ is

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = 0$$

$$\Rightarrow \frac{d}{dt} (m_2 l^2 \ddot{\theta} + m_2 \dot{x} l \cos \theta) + m_2 l \dot{x} \dot{\theta} \sin \theta + m_2 g l \sin \theta = 0$$

$$\Rightarrow m_2 l^2 \ddot{\theta} + m_2 l \cos \theta \ddot{x} + m_2 g l \sin \theta = 0 \quad \dots(3)$$

The Lagrangian does not contain coordinate x , so it is cyclic, the corresponding momentum is constant, so

$$p_x = \frac{\partial L}{\partial \dot{x}} = (m_1 + m_2) \dot{x} + m_2 l \dot{\theta} \sin \theta = \text{constant}$$

For simplicity this constant can be taken as zero.

$$\text{then } \dot{x} = - \frac{m_2 l \dot{\theta} \cos \theta}{m_1 + m_2}$$

$\therefore$ Equation (3) becomes

$$m_2 l^2 \ddot{\theta} - m_2 l \cos \theta \cdot \left(\frac{m_2 l}{m_1 + m_2} \right) \cdot (\cos \theta \ddot{\theta} - \sin \theta \dot{\theta}^2) + m_2 g l \sin \theta = 0$$

$$\Rightarrow \ddot{\theta} - \frac{m_2 l}{m_1 + m_2} (\ddot{\theta} \cos^2 \theta - \cos \theta \sin \theta \dot{\theta}^2) + g \sin \theta = 0 \quad \dots(5)$$

For small oscillations θ is very small

So $\cos \theta \approx 1$ and $\sin \theta \approx \theta$

In view of this, equation (5) becomes

$$l \ddot{\theta} - \frac{m_2 l}{m_1 + m_2} (\ddot{\theta} - \theta \dot{\theta}^2) + g \theta = 0$$

$$\Rightarrow l \ddot{\theta} - \frac{m_2 l}{m_1 + m_2} \ddot{\theta} + g \theta = 0 \text{ (neglecting higher order terms in } \theta)$$

$$\Rightarrow \ddot{\theta} \left(l - \frac{m_2 l}{m_1 + m_2} \right) + g \theta = 0$$

$$\Rightarrow \ddot{\theta} + \frac{m_1 + m_2}{m_1} \frac{g}{l} \theta = 0 \quad \dots(6)$$

Comparing with standard equations $\ddot{\theta} + \omega^2 \theta = 0$, we get

$$\text{Angular frequency } \omega = \sqrt{\frac{m_1 + m_2}{m_1} \left(\frac{g}{l} \right)}$$

$$\therefore \text{Linear frequency } f = \frac{\omega}{2\pi} = \frac{1}{2\pi} \sqrt{\left\{ \frac{g}{l} \left(\frac{m_1 + m_2}{m_1} \right) \right\}}$$

15.15. The Mechanics of a Rigid Body

15.15. (a) Displacement of a Rigid Body. A rigid body may be defined as a system of particles whose mutual distances remain all fixed. The internal forces holding the particles at fixed distances from one another or more exactly the forces maintaining certain fixed relations between the particles of a system (rigid body) are known as *forces of constraint*. These forces of constraint obey Newton's third law of motion and therefore the laws of conservation of linear and angular momentum can be applied to the motion of a rigid body.

While considering the motion of a rigid body in three dimensional space, if we fix up a point of it, the body is free to turn about every line through this point. Again, if we fix up a second point also then the motion of the body is restricted to rotate about the line joining the two fixed points. Further, if we also fix up a third point of the body non-collinear with other two, then the position of the body is fixed. It follows that the position of a body is determined by the positions in space of any three non-collinear points of it and the positions of three points are determined by nine co-ordinates. connected by three relations which express the distances between the three chosen points of the body, so that there are only six independent variables or *degrees of freedom*.

Let $\hat{i}, \hat{j}, \hat{k}$ be the vectors along the axes of reference chosen with respect to some arbitrary fixed origin. If the origin remaining fixed, the axes are rotated in directions along which the direction cosines are $(l_1, m_1, n_1), (l_2, m_2, n_2), (l_3, m_3, n_3)$ respectively such that $\hat{i}', \hat{j}', \hat{k}'$ are the unit vectors along these directions, then we have from the notions of vectors.

$$\left. \begin{aligned} \hat{i}' &= l_1 \hat{i} + m_1 \hat{j} + n_1 \hat{k}, \\ \hat{j}' &= l_2 \hat{i} + m_2 \hat{j} + n_2 \hat{k}, \\ \hat{k}' &= l_3 \hat{i} + m_3 \hat{j} + n_3 \hat{k}, \end{aligned} \right\} \quad (1)$$

since $\hat{i}' = (\hat{i}' \cdot \hat{i}) \hat{i} + (\hat{i}' \cdot \hat{j}) \hat{j} + (\hat{i}' \cdot \hat{k}) \hat{k}$ etc. where $\hat{i}' \cdot \hat{i} = \cos(\hat{i}' \cdot \hat{i}) = l_1$ etc, the nine coefficients $l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3$ are related by the six equations.

$$\left. \begin{aligned} 1 &= \hat{i}' \cdot \hat{i}' = \hat{i}'^2 = l_1^2 + m_1^2 + n_1^2 \\ 1 &= \hat{j}' \cdot \hat{j}' = \hat{j}'^2 = l_2^2 + m_2^2 + n_2^2 \\ 1 &= \hat{k}' \cdot \hat{k}' = \hat{k}'^2 = l_3^2 + m_3^2 + n_3^2 \end{aligned} \right\} \quad \dots(2)$$

since $\hat{i}'^2 = (l_1 \hat{i} + m_1 \hat{j} + n_1 \hat{k}) \cdot (l_1 \hat{i} + m_1 \hat{j} + n_1 \hat{k}) = l_1^2 + m_1^2 + n_1^2$ as $\hat{i} \cdot \hat{i} = 1$
and $\hat{i} \cdot \hat{j} = 0$ etc,

$$\left. \begin{aligned} 0 &= \hat{i}' \cdot \hat{j}' = l_1 l_2 + m_1 m_2 + n_1 n_2 \\ 0 &= \hat{j}' \cdot \hat{k}' = l_2 l_3 + m_2 m_3 + n_2 n_3 \\ 0 &= \hat{k}' \cdot \hat{i}' = l_3 l_1 + m_3 m_1 + n_3 n_1 \end{aligned} \right\} \quad \dots(3)$$

since $\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1$ and $\hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0$.

The six equations given by (2) and (3) show that there are nine co-ordinates but the number of independent coordinates are replaced to only six and hence the position of a rigid body can be determined by six numbers, *i.e.*, the number of degrees of freedom is six.

The number of degrees of freedom is reduced by constraints. For example, if only one point of the rigid body is fixed, then placing the origin of co-ordinates at that point we observe that the displaced position of the body can be determined by three numbers only and hence the degree of freedom is then three. If we hold two points fixed, the only possible motion is rotation about the line joining the two fixed points and therefore the position is then completely determined by giving the angle of rotation and hence in this case the degree of freedom is one.

In general, a rigid body with N particles, free from constraints can at most have $3N$ degrees of freedom, which are reduced by constraints given by

$$r_{\alpha\beta} = a_{\alpha\beta} \quad (\alpha, \beta = 1, 2, 3, \dots, v, \alpha \neq \beta) \quad \dots(4)$$

where $r_{\alpha\beta}$ denotes the distance between α th and β th particles and a are constants

The possible forms of equations (4) are

$$\begin{aligned} &= {}^NC_2 \\ &= \frac{N(N-1)}{2} > 3N \text{ for } N > 6. \end{aligned}$$

$\therefore$ The actual number of degrees of freedom can be obtained simply by subtracting the number of constraint equations from $3N$.

If we take $N = 3$, then the number of constraint equations $= \frac{1}{2} N(N-1) = \frac{3}{2} \cdot 2 = 3$ which when subtracted from $3N (=9)$ gives the degrees of freedom as 6 which is in accordance with the above result. In fact, the equations (4) are not all independent. Conclusively, to fix a point in the rigid body, its distance from all other points should not be specified, but its distance from any three other collinear points are needed; for, the positions three particles being determined, the constraints fix the positions of all the remaining particles. As such the number of degrees of freedom cannot exceed nine. Actually, the three reference points not being independent, the three equations of rigid constraints i.e.,

$$r_{12} = a_{12}, r_{23} = a_{23}, r_{13} = a_{13}$$

reduce the number of degrees of freedom to six. Hence **a rigid body consisting of N number of particles requires six independent generalized coordinates for the specification of its configuration.**

The equations (2) and (3) can be symbolically written as

$$\left. \begin{aligned} l_{\alpha}^2 + m_{\alpha}^2 + n_{\alpha}^2 &= 1, \quad \alpha = 1, 2, 3 \\ l_{\alpha}l_{\beta} + m_{\alpha}m_{\beta} + n_{\alpha}n_{\beta} &= 0, \quad \alpha, \beta = 1, 2, 3; \alpha \neq \beta \end{aligned} \right\} \quad \dots(5)$$

These six equations given by (5) can be combined into one by using Kronecker δ -symbol as

$$l_{\alpha}l_{\beta} + m_{\alpha}m_{\beta} + n_{\alpha}n_{\beta} = \delta_{\alpha\beta} \quad \dots(6)$$

where $\delta_{\alpha\beta} = 1$ when $\alpha = \beta$

$= 0$ when $\alpha \neq \beta$.

If we now assume that the sets of nine direction cosines $(l_{\alpha}, m_{\alpha}, n_{\alpha})$, $\alpha = 1, 2, 3$, completely specify the orientation of the x', y', z' axes relative to the x, y, z set, then using the vector notions we easily get

$$\begin{aligned} \hat{i} &= (\hat{i}, \hat{i}') \hat{i}' + (\hat{i}, \hat{j}') \hat{j}' + (\hat{i}, \hat{k}') \hat{k}' \\ &= l_1 \hat{i}' + m_1 \hat{j}' + n_1 \hat{k}' \text{ as } \hat{i} \cdot \hat{i}' = \cos(\hat{i}, \hat{i}') = l_1 \text{ etc.} \end{aligned}$$

with similar expressions for $\hat{j}$ and $\hat{k}$.

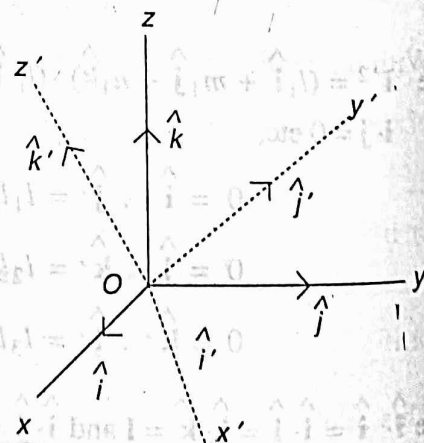


Fig. 15-32

Now if $\mathbf{r}$ represents the position vector of a point whose coordinates are (x, y, z) so that

$$\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}.$$

we have $x' = \mathbf{r} \cdot \hat{\mathbf{i}}' = (x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}) \cdot \hat{\mathbf{i}}'$ as $x' = r \cos(\hat{\mathbf{i}}, \hat{\mathbf{i}}')$

$$= x(\hat{\mathbf{i}} \cdot \hat{\mathbf{i}}') + y(\hat{\mathbf{j}} \cdot \hat{\mathbf{i}}') + z(\hat{\mathbf{k}} \cdot \hat{\mathbf{i}}')$$

$$\text{i.e., and similarly } \left. \begin{aligned} x' &= l_1x + l_2y + l_3z \\ y' &= m_1x + m_2y + m_3z \\ z' &= n_1x + n_2y + n_3z \end{aligned} \right\} \dots(7)$$

Thus the transformation between the coordinate-system is affected by the set of nine direction cosines and to specify the orientation only three coordinates are needed.

15.14 (b) Orthogonal Transformations : For the sake of convenience if we denote all coordinates by x , the subscripts giving the axes such that $x \rightarrow x_1, y \rightarrow x_2, z \rightarrow x_3$, then the equations (7) of § 15.12 (a) can be written as

$$\left. \begin{aligned} x_1' &= l_1x_1 + l_2x_2 + l_3x_3 \\ x_2' &= m_1x_1 + m_2x_2 + m_3x_3 \\ x_3' &= n_1x_1 + n_2x_2 + n_3x_3 \end{aligned} \right\} \dots(1)$$

These equations form a linear or vector transformation defined by the transformations of the type

$$\left. \begin{aligned} x_1' &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ x_2' &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ x_3' &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{aligned} \right\} \dots(2)$$

where $a_{\alpha\beta}$ ($\alpha, \beta = 1, 2, 3; \alpha \neq \beta$) are constants.

The three equations of (3) can be combined into one as

$$x_\alpha' = \sum_{\beta=1}^3 a_{\alpha\beta} x_\beta, \quad \alpha = 1, 2, 3. \dots(4)$$

Now we know that the actual vectors remain unchanged irrespective of the coordinate system; therefore in order that the magnitude of the vector is invariant in both the systems of co-ordinates, we must have

$$\sum_{\alpha=1}^3 x_\alpha'^2 = \sum_{\alpha=1}^3 x_\alpha^2. \dots(5)$$

With the help of (4), (5) becomes

$$\sum_{\alpha=1}^3 \left(\sum_{\beta=1}^3 a_{\alpha\beta} x_\beta \right) \left(\sum_{\gamma=1}^3 a_{\alpha\gamma} x_\gamma \right) = \sum_{\alpha=1}^3 x_\alpha^2$$

$$\text{or if } \sum_{\alpha=1}^3 \left(\sum_{\beta, \gamma=1}^3 a_{\alpha\beta} (a_{\alpha\gamma}) x_\beta x_\gamma \right) = \sum_{\alpha=1}^3 x_\alpha^2$$

$$\text{or if } \sum_{\beta=1}^3 \left(\sum_{\alpha=1}^3 a_{\alpha\beta} a_{\alpha\gamma} \right) x_\beta x_\gamma = \sum_{\alpha=1}^3 x_\alpha^2$$

This relation will hold good only if

$$\sum_{\alpha=1}^3 a_{\alpha\beta} a_{\alpha\gamma} = 1 \text{ when } \beta = \gamma; \\ = 0 \text{ when } \beta \neq \gamma$$

or this may be expressed as

$$\sum_{\alpha=1}^3 a_{\alpha\beta} a_{\alpha\gamma} = \delta_{\beta\gamma}, \quad \beta, \gamma = 1, 2, 3. \quad \dots(6)$$

Any linear transformation having this property is known as *orthogonal transformation* and the equations (6) are sometimes called *orthogonality conditions*.

An array containing transformation constants, i.e.,

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \text{ or } \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \text{ or } \left\| \begin{matrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{matrix} \right\|$$

is termed as the *matrix of transformation* and denoted by *A* i.e.,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \text{ or in short } A = [a_{ij}]. \quad \dots(7)$$

the quantities a_{ij} comprising the matrix are called the *matrix elements*.

Here *A* sometimes acts as an operator to transform unprimed axes into primed axes such that $(\mathbf{r})' = \mathbf{A}\mathbf{r}$

... (8)

read as 'the matrix *A* operating on the components of a vector in the unprimed system, yields the components of the vector in the primed system.'

In case of two-dimensional transformation of co-ordinates, simply the rotation takes place and so *A* is known as the *rotation operator* in a plane. It can be shown as below :

In case of motion in a plane, the indices α and β entered in (6) take values 1, 2 only and so (6) becomes

$$\sum_{\alpha=1}^2 a_{\alpha\beta} a_{\alpha\gamma} = \delta_{\beta\gamma}, \quad \beta, \gamma = 1, 2$$

and then (7) yields

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

Now we know from co-ordinate geometry that if origin remains fixed, the axes x_1', x_2' rotated through an angle $-\theta$ to take the positions x_1, x_2 , then transformation equations are

$$x_1' = x_1 \cos(-\theta) - x_2 \sin(-\theta) = x_1 \cos \theta + x_2 \sin \theta.$$

$$x_2' = x_1 \sin(-\theta) + x_2 \cos(-\theta) = -x_1 \sin \theta + x_2 \cos \theta.$$

so that, in view of (3), the matrix elements are

$$a_{11} = \cos \theta, a_{12} = \sin \theta, a_{21} = -\sin \theta, a_{22} = \cos \theta.$$

Thus

$$A = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$$

and the matrix elements satisfy the orthogonality conditions

$$a_{11}a_{11} + a_{21}a_{21} = 1,$$

$$a_{12}a_{12} + a_{22}a_{22} = 1,$$

$$a_{11}a_{12} + a_{21}a_{22} = 0,$$

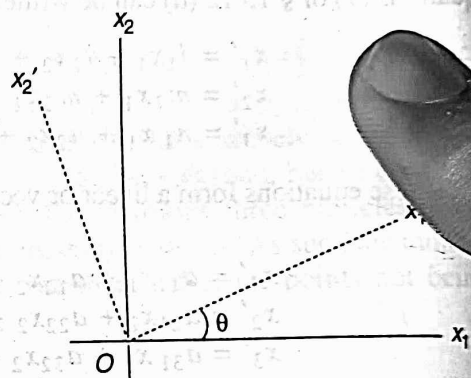


Fig. 15-33

$$\text{i.e., } \left. \begin{aligned} \cos^2 \theta + \sin^2 \theta &= 1, \\ \sin^2 \theta + \cos^2 \theta &= 1, \\ \cos \theta \sin \theta - \sin \theta \cos \theta &= 0. \end{aligned} \right\} \dots(12)$$

Hence in case of the motion in a plane, A acts as rotation operator.

Further, expressing both the vectors in the same co-ordinate system A can be regarded as an operator acting on $\mathbf{r}$ and changing it to a different sector $\mathbf{r}'$. so that (8) can be written as

$$\mathbf{r}' = A\mathbf{r}$$

15.15 (c) Eulerian angles : In place of nine coefficients conditioned by six equations discussed in § 15.15 (a) the three independent parameters (variables) can be used to specify the orientation of a rigid body. The transformation from a given cartesian coordinate system to another can be affected by describing the rotation of a rigid body about a point in terms of three successive rotations given in the following order.

(i) Rotation of the initial system of axes x, y, z through ϕ counterclockwise about the z -axis and let the resultant coordinate system be labelled as ξ, η, ζ axes.

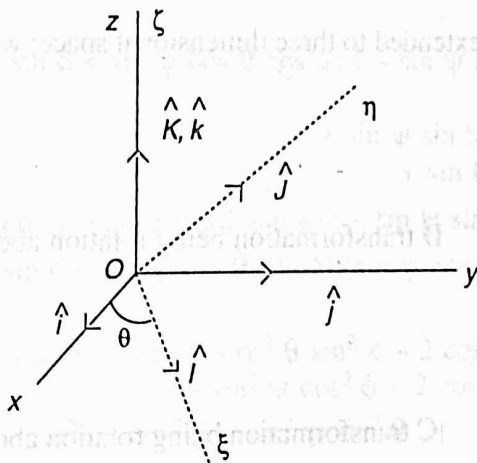


Fig. 15-34

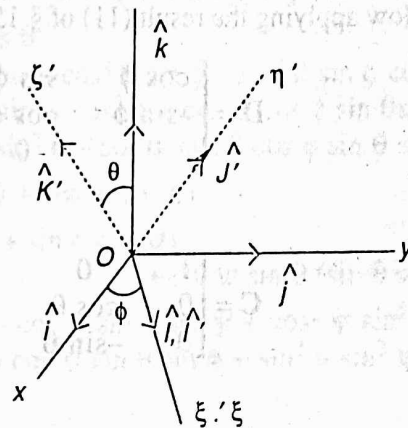


Fig. 15-35

(ii) Rotation of the axes ξ, η, ζ about the ξ -axis through an angle θ counterclockwise and let the resultant set be relabelled as ξ', η', ζ' axes. Here ξ' -axis being the line of intersection of x - y and ξ' - η' planes is called the *line of nodes*.

(iii) Rotation of the axes ξ', η', ζ' through an angle ψ about the ζ' axis to give finally the x', y', z' system of axes.

Here x, y, z is the old set of axes and x', y', z' is the new set of axes obtained after rotations (i), (ii), (iii). The angles θ, ϕ, ψ are known as *Eulerian angles*, which completely specify the orientation of the x', y', z' system relative to x, y, z system and act as three generalised coordinates.

Let A be the complete matrix of transformation from x, y, z to x', y', z' ; D the matrix of transformation from x, y, z to ξ, η, ζ ; C that from ξ, η, ζ to ξ', η', ζ' and B that from ξ', η', ζ' to x', y', z' . Then if $\vec{\xi}$ and $\mathbf{x}$ represent the column matrices, we have

$$\vec{\xi} = D\mathbf{x}$$

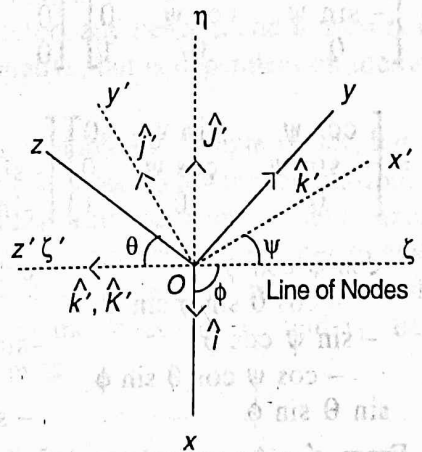


Fig. 15-36

Similarly if $\vec{\xi}'$ and x' represent the column matrices, we have

$$\vec{\xi}' = C\vec{\xi}$$

and

$$x' = B\vec{\xi}'$$

while

$$x' = Ax$$

so that

$$Ax = x' = B\vec{\xi}'$$

$$= BC\vec{\xi}$$

$$= BCDx$$

giving

$$A = BCD$$

i.e., the complete matrix of transformation A is the product of the successive matrices transformations DCB .

Now applying the result (11) of § 15-10(b), extended to three dimensional space; we have

$$D = \begin{bmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

D transformation being rotation about z -axis

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & \sin \theta \\ 0 & -\sin \theta & \cos \theta \end{bmatrix}$$

C transformation being rotation about ξ -axis

and

$$B = \begin{bmatrix} \cos \psi & \sin \psi & 0 \\ -\sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

B transformation being rotation about ξ' -axis

Making substitutions from (2), (3) and (4) in (1), we get

$$A = \begin{bmatrix} \cos \psi & \sin \psi & 0 \\ -\sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & \sin \theta \\ 0 & -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos \psi & \sin \psi & 0 \\ -\sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \theta \sin \phi & +\cos \theta \cos \phi & \sin \theta \\ \sin \theta \sin \phi & -\sin \theta \cos \phi & \cos \theta \end{bmatrix}$$

$$\begin{bmatrix} \cos \psi \cos \phi & \cos \psi \sin \phi & \sin \psi \sin \theta \\ -\cos \theta \sin \phi \sin \psi & +\cos \theta \cos \phi \sin \psi & \cos \psi \sin \theta \\ -\sin \psi \cos \phi & -\sin \psi \sin \phi & \cos \psi \sin \theta \\ -\cos \psi \cos \theta \sin \phi & +\cos \psi \cos \theta \cos \phi & \cos \theta \\ \sin \theta \sin \phi & -\sin \theta \cos \phi & \cos \theta \end{bmatrix} \quad (5)$$

From $x' = Ax$ we get $x = A^{-1}x'$.

which gives the inverse transformation from body co-ordinates to space axes and the matrix of transformation is then $A^{-1} = A^T$ in case of orthogonal transformation matrix.

$$\begin{bmatrix}
 \cos \psi \cos \phi & -\sin \psi \cos \phi & \sin \theta \sin \phi \\
 -\cos \theta \sin \phi \sin \psi & -\cos \theta \sin \phi \cos \psi & \\
 \cos \psi \sin \phi & -\sin \psi \sin \phi & -\sin \psi \cos \phi \\
 +\cos \theta \cos \phi \sin \psi & +\cos \theta \cos \phi \cos \psi & \\
 \sin \theta \sin \psi & \sin \theta \cos \psi & \cos \theta
 \end{bmatrix} \quad \dots(6)$$

To be sure that A is an orthogonal matrix we can show that

$$|A| = \pm 1.$$

Here

$$|A| = \begin{vmatrix}
 \cos \psi \cos \phi & -\sin \psi \cos \phi & \sin \theta \sin \phi \\
 -\sin \psi \cos \theta \sin \phi & +\sin \psi \cos \theta \cos \phi & \sin \psi \sin \theta \\
 -\sin \psi \cos \phi & -\sin \psi \sin \phi & -\sin \psi \cos \phi \\
 -\cos \psi \cos \theta \sin \phi & +\cos \psi \cos \theta \cos \phi & \cos \psi \sin \theta \\
 \sin \theta \sin \phi & -\sin \theta \cos \phi & \cos \theta
 \end{vmatrix}$$

Expanding the R.H.S. determinant about first row, we get

$$\begin{aligned}
 |A| &= (\cos \psi \cos \phi - \sin \psi \cos \theta \sin \phi) \{ -\sin \psi \sin \phi \cos \theta \\
 &\quad + \cos \psi \cos^2 \theta \cos \phi + \cos \psi \sin^2 \theta \cos \phi \} \\
 &\quad - (\cos \psi \sin \phi + \sin \psi \cos \theta \cos \phi) \{ -\sin \psi \cos \phi \cos \theta \\
 &\quad \quad - \cos \psi \cos^2 \theta \sin \phi - \sin^2 \theta \sin \phi \cos \psi \} \\
 &\quad + \sin \psi \sin \theta \{ \sin \theta \sin \psi \cos^2 \phi + \cos \psi \cos \phi \sin \theta \sin \phi \} \\
 &\quad + \sin \theta \sin \psi \sin^2 \phi - \cos \psi \cos \theta \cos \phi \sin \theta \sin \phi \} \\
 &= (\cos \psi \cos \phi - \sin \psi \cos \theta \sin \phi) (-\sin \psi \sin \phi \cos \theta + \cos \psi \cos \phi) \\
 &\quad + (\cos \psi \sin \phi + \sin \psi \cos \theta \cos \phi) (\sin \psi \cos \phi \cos \theta + \sin \phi \cos \psi) \\
 &\quad + \sin \psi \sin \theta (\sin \theta \sin \psi) \\
 &= \cos^2 \psi \cos^2 \phi + \sin^2 \psi \cos^2 \theta \sin^2 \phi - 2 \cos \theta \cos \phi \cos \psi \sin \phi \sin \psi + \cos^2 \psi \sin^2 \phi \\
 &\quad + \sin^2 \psi \cos^2 \phi + 2 \cos \theta \cos \phi \cos \psi \sin \theta \sin \phi + \sin^2 \theta \sin^2 \psi \\
 &= \cos^2 \psi + \sin^2 \psi \cos^2 \theta + \sin^2 \psi \sin^2 \theta \\
 &= \cos^2 \psi + \sin^2 \psi \\
 &= 1.
 \end{aligned}$$

15-15 (d) Infinitesimal Rotation. In order to see whether a vector can be associated with the finite rotation represented by an orthogonal transformation, let us consider two such vectors, a and b associated with matrices of transformation A and B . Now we know that the resultant of two successive rotations is given by the product AB of the two matrices of transformations. Since the matrix product is not commutative, *i.e.*, $AB \neq BA$, it follows that the addition (resultant) of associated vectors a and b is not commutative, *i.e.*, $a + b \neq b + a$. But this commutation in addition is a characteristic property of vectors and hence a and b must not be vectors. As such the sum of finite rotations is not commutative, but is dependent on the order of the rotations.

We have thus shown that a finite rotation is not re-presentable by a single vector, but this is not the case with the infinitesimal rotations as is shown in the following discussion. An infinitesimal rotation may be regarded as an orthogonal transformation from one set of axes to another such that the components of a vector are the same in both the sets in order to ensure that the change is infinitesimal. Thus, if x' be a component of vector r' in the new set while its value in the old set is x_1 then the difference between the values of x_1' and x_1 being infinitely small, we can express a relationship between them as

$$x' = x_1 = q_{11} x_1 + q_{12} x_2 + q_{13} x_3. \quad \dots(1)$$

where the matrix elements q_{11}, q_{12}, q_{13} are very small.

In general, the relation (1) can be written as

$$x'_\alpha = x_\alpha + \sum_{\beta} q_{\alpha\beta} x_\beta = \sum_{\beta} (\delta_{\alpha\beta} + q_{\alpha\beta}) x_\beta \quad \dots(2)$$

where $\delta_{\alpha\beta} = 1$ for $\alpha = \beta$ and $= 0$ for $\alpha \neq \beta$, so that $\delta_{\alpha\beta}$ may be recognised as the element of a unit matrix. Hence applying the matrix rotations, we can write (2) as

$$\mathbf{x}' = (\mathbf{I} + \mathbf{q}_2) \mathbf{x} \quad \dots(3)$$

This shows that the matrix of transformation for infinitesimal rotations is of the form $\mathbf{I} + \mathbf{q}$. Hence if $\mathbf{I} + \mathbf{q}_1$ and $\mathbf{I} + \mathbf{q}_2$ are two matrices of infinitesimal transformations, then

$$(\mathbf{I} + \mathbf{q}_1)(\mathbf{I} + \mathbf{q}_2) = \mathbf{I}^2 + \mathbf{q}_1\mathbf{I} + \mathbf{I}\mathbf{q}_2 + \mathbf{q}_1\mathbf{q}_2 = \mathbf{I} + \mathbf{q}_1 + \mathbf{q}_2, \quad \dots(4)$$

neglecting the product $\mathbf{q}_1\mathbf{q}_2$.

$$\text{Also } (\mathbf{I} + \mathbf{q}_2)(\mathbf{I} + \mathbf{q}_1) = \mathbf{I}^2 + \mathbf{q}_2\mathbf{I} + \mathbf{I}\mathbf{q}_1 + \mathbf{q}_2\mathbf{q}_1 = \mathbf{I} + \mathbf{q}_2 + \mathbf{q}_1 \quad \dots(5)$$

Since the matrix addition is commutative, i.e., $\mathbf{q}_1 + \mathbf{q}_2 = \mathbf{q}_2 + \mathbf{q}_1$ it therefore, follows from (4) and (5) that the product of infinitesimal transformations is commutative.

$$\text{If } \mathbf{A} = \mathbf{I} + \mathbf{q}, \text{ then } \mathbf{A}^{-1} = (\mathbf{I} + \mathbf{q})^{-1} = \mathbf{I} - \mathbf{q}, \quad \dots(6)$$

since $\mathbf{I} = \mathbf{A}\mathbf{A}^{-1} = (\mathbf{I} + \mathbf{q})(\mathbf{I} - \mathbf{q})$

$$= \mathbf{I} \text{ neglecting the product } \mathbf{q}^2.$$

Consider a specific infinitesimal rotation about z-axis. In case of finite rotation, the transformation matrix $\mathbf{A}$ is given by (2) of § 15-15c as

$$\mathbf{A} = \begin{bmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

When the rotation is infinitesimal, then

$$\cos \delta\phi \rightarrow 1 \text{ and } \sin \delta\phi \rightarrow \delta\phi,$$

so that we have

$$\mathbf{I} + \mathbf{q} = \begin{bmatrix} 1 & d\phi & 0 \\ -d\phi & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{q} = \begin{bmatrix} 1 & d\phi & 0 \\ -d\phi & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ taking } \mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & d\phi & 0 \\ -d\phi & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = d\phi \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \dots(7)$$

This shows that $\mathbf{q}$ is a skew-symmetric matrix, because $\mathbf{q} = -\mathbf{q}^T$ giving $q_{\alpha\beta} = q_{\beta\alpha}$.

Another method of explanation. Suppose that a rigid body with a fixed point O is turned through an infinitesimal angle about an axis through O and let the infinitesimal rotation be represented by an infinitesimal vector $\delta\mathbf{n}$. Let $\mathbf{r}$ and $\mathbf{r} + \delta\mathbf{r}$ be the position vectors of particle A before and after the displacement. Let AN be the perpendicular from A to $\delta\mathbf{n}$ and let $\angle AON = \theta$ so that $AN = OA \sin \theta = r \sin \theta$ where r is magnitude of $\mathbf{r}$.

Now we know that the vector product (of vectors $\delta\mathbf{n}$ and $\mathbf{r}$) $\delta\mathbf{n} \times \mathbf{r}$ is a vector perpendicular to the plane of $\delta\mathbf{n}$ and $\mathbf{r}$; also the magnitude of the displacement $\delta\mathbf{r}$ is given by

$$\delta\mathbf{r} = AN. \delta\mathbf{n} = r \sin \theta \delta\mathbf{n}.$$

$$\therefore \delta\mathbf{r} = \delta\mathbf{n} \times \mathbf{r} \quad \dots(8)$$

by the definition of cross product as well as the directions of the magnitudes of $\delta\mathbf{n} \times \mathbf{r}$ being the same

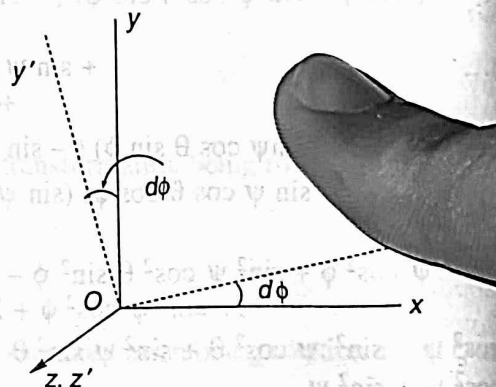


Fig. 15-37

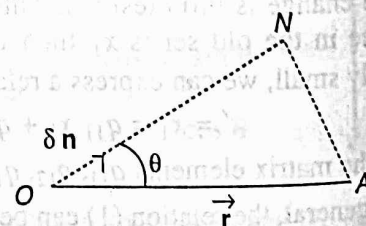


Fig. 15-38

In order to find the resultant of two successive infinitesimal rotations $\delta \mathbf{n}$ and $\delta \mathbf{n}'$ about an axis through O , we get on applying the result (1) the position vector of A as.

(i) $\mathbf{r}$ before displacement (rotation),

(ii) $\mathbf{r} + \delta \mathbf{n} \times \mathbf{r}$ after the first rotation,

(iii) $\mathbf{r} + \delta \mathbf{n} \times \mathbf{r} + \delta \mathbf{n}' \times (\mathbf{r} + \delta \mathbf{n} \times \mathbf{r})$, after second rotation,

Thus the resultant displacement

$$= \{\mathbf{r} + \delta \mathbf{n} \times \mathbf{r} + \delta \mathbf{n}' \times (\mathbf{r} + \delta \mathbf{n} \times \mathbf{r})\} - \mathbf{r}$$

$$= \{\mathbf{r} + \delta \mathbf{n} \times \mathbf{r} + \delta \mathbf{n}' \times \mathbf{r}\} - \mathbf{r}, \text{ neglecting the infinitesimal } \delta \mathbf{n}' \times \delta \mathbf{n} \times \mathbf{r}$$

$$= \delta \mathbf{n} \times \mathbf{r} + \delta \mathbf{n}' \times \mathbf{r}$$

$$= (\delta \mathbf{n} + \delta \mathbf{n}') \times \mathbf{r}$$

$$= \text{displacement of } A \text{ from a single rotation } (\delta \mathbf{n} + \delta \mathbf{n}').$$

This follows that the resultant of two infinitesimal rotations about the same point is the vector sum of these rotations.

15.15 (e) Kinematics of a Rigid Body.

Motion of a rigid body about a fixed point. Let a rigid body be constrained to rotate about a fixed point O and let its displacement in time interval Δt from t_1 to t_2 be equivalent to a rotation $\delta \mathbf{n}$ about O . Keeping t_1 fixed and $t_2 \rightarrow t_1$ let the limiting of direction $\mathbf{n}$ be denoted by $\hat{\mathbf{i}}$.

$$\text{Assuming that } \frac{\delta \mathbf{n}}{\Delta t} \rightarrow \vec{\omega}, \text{ we have } \vec{\omega} = \omega \hat{\mathbf{i}}. \quad \dots(1)$$

Here $\vec{\omega}$ is known as the *angular velocity* of the body at time t_1 . At t_1 the body is rotating about a line through O in the direction of $\vec{\omega}$; this is known as the *instantaneous axis of rotation*.

The infinitesimal rotation of the body in an infinitesimal time dt is ωdt so that by (8) of § 15.14 (d) the displacement of a particle of the body is given by

$$d\mathbf{r} = \vec{\omega} dt \times \mathbf{r}. \quad \dots(2)$$

$\mathbf{r}$ being the position vector of the particle relative to O .

$\therefore$ The velocity of this particle is given by

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \vec{\omega} \times \mathbf{r} \quad \dots(3)$$

With the rotation of the body O , the instantaneous axis occupies different position in the body but always passing through O . The locus of this axis in the body is a cone with vertex O known as *body cone* or *polhode cone*. Similarly, its locus in space is a cone with vertex O known as *space cone* or *herpolhode cone*.

If the general motion of the body is considered, then taking a particle A as base point with velocity $\mathbf{v}_A$, the displacement of the body in an infinitesimal time dt is equivalent to a motion of translation $\mathbf{v}_A dt$ and a motion of rotation $d\mathbf{n}$ about A so that the displacement of another particle B is

$$\mathbf{v}_A dt + d\mathbf{n} \times \mathbf{r} \quad \text{where } \mathbf{r} = \vec{AB}$$

Hence taking $\vec{\omega} = \frac{d\mathbf{n}}{dt}$, the velocity of B is given by

$$\mathbf{v} = \mathbf{v}_A + \vec{\omega} \times \mathbf{r}, \quad \dots(4)$$

i.e., Velocity of B = Velocity of base point A + Velocity of B relative to A .

Acceleration of B can be given by

$$\begin{aligned} \mathbf{a} &= \frac{d\mathbf{v}}{dt} = \frac{d\mathbf{v}_A}{dt} + \frac{d\vec{\omega}}{dt} \times \mathbf{r} + \vec{\omega} \times \frac{d\mathbf{r}}{dt} \\ &= \mathbf{a}_A + \frac{d\vec{\omega}}{dt} \times \mathbf{r} + \vec{\omega} \times (\vec{\omega} \times \mathbf{r}) \end{aligned} \quad \dots(5)$$

since acceleration of $A = \mathbf{a}_A = \frac{d\mathbf{v}_A}{dt}$ and $\frac{d\mathbf{r}}{dt} = \vec{\omega} \times \mathbf{r}$ (6)

15.15 (f) Moments and Products of Inertia. We know that the *moment of inertia* of a particle of mass m about a line OL , distance p from OL is given by mp^2 . Now to define *product of inertia*, consider two planes P_1 and P_2 at distances p_1 and p_2 respectively from a particle of mass m ; then the product mp_1p_2 is known as the *product of inertia* of the particle with respect to the planes P_1 and P_2 . For a rigid body regarded as a system of particles the moments and products of inertia are the sum of the moments and products of inertia of the several particles. Hence for a rigid body having a representative particle of mass m situated at (x, y, z) with respect to a rectangular axis Ox, Oy, Oz . If A, B, C be the moments of inertia about Ox, Oy, Oz and F, G, H the products of inertia with respect to the co-ordinate planes, taken two in pairs we have

$$A = \sum m (y^2 + z^2), B = \sum m (z^2 + x^2), C = \sum m (x^2 + y^2). \quad \dots(7)$$

$$F = \sum myz, G = \sum mzx, H = \sum mxy. \quad \dots(8)$$

In order to find the moment of inertia I of a system about any line OL through O , when A, B, C, F, G, H are known, take a particle of mass m distant

p from OL . Then, if $\angle POL = \theta$, $\vec{OP} = \mathbf{r}$ and $\hat{\mathbf{l}}$ be a unit vector along OL , we have

$$p = OP \sin \theta$$

$$= r \sin \theta$$

$$= |\mathbf{r}| \sin \theta = |\hat{\mathbf{l}} \parallel \mathbf{r}| \sin \theta \quad [\text{since } |\hat{\mathbf{l}}| = 1]$$

$= |\hat{\mathbf{l}} \times \mathbf{r}|$ by the definition of cross product of two vectors.

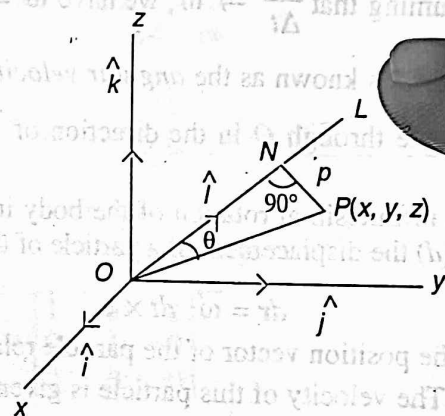


Fig. 15-39

Let (x, y, z) be the coordinates of P and λ, μ, ν the direction cosines of OL ; then taking $\mathbf{i}, \mathbf{j}, \mathbf{k}$ triad of unit vector in the directions of x, y, z axes, we have

$$\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} \text{ and } \hat{\mathbf{l}} = \lambda\hat{\mathbf{i}} + \mu\hat{\mathbf{j}} + \nu\hat{\mathbf{k}}.$$

$$\text{so that } \hat{\mathbf{l}} \times \mathbf{r} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \lambda & \mu & \nu \\ x & y & z \end{vmatrix} = (\mu z - \nu y) \hat{\mathbf{i}} + (\nu x - \lambda z) \hat{\mathbf{j}} + (\lambda y - \mu x) \hat{\mathbf{k}}.$$

$$\therefore |\hat{\mathbf{l}} \times \mathbf{r}| = p = \sqrt{[(\mu z - \nu y)^2 + (\nu x - \lambda z)^2 + (\lambda y - \mu x)^2]}$$

$$\text{Hence } I = \sum mp^2$$

$$= \sum m \{(\mu z - \nu y)^2 + (\nu x - \lambda z)^2 + (\lambda y - \mu x)^2\}$$

$$= \lambda^2 \sum m (y^2 + z^2) + \mu^2 \sum m (z^2 + x^2) + \nu^2 \sum m (x^2 + y^2)$$

$$- 2\mu\nu \sum myz - 2\nu\lambda \sum mzx - 2\lambda\mu \sum mxy$$

$$= A\lambda^2 + B\mu^2 + C\nu^2 - 2F\mu\nu - 2G\nu\lambda - 2H\lambda\mu \quad \dots(8)$$

by equations (6) and (7).

which gives I in terms of A, B, C, F, G, H and λ, μ, ν .

From (8) it is obvious that for different values of λ, μ, ν we can get the moments of inertia about all lines passing through O . If we measure off, along each line through O , a distance $OQ = 1/\sqrt{I}$, where I is the moment of inertia about the line under consideration, then the locus of Q is found to be

$$Ax^2 + By^2 + Cz^2 - 2Fyz - 2Gzx - 2Hxy = 1 \quad \dots(9)$$

which is called the *ellipsoid of inertia* or the *momental ellipsoid* at O since the quadratic (9) represents an ellipsoid.

The geometrical property of a momental ellipsoid is that any of its radius vectors is equal to some constant divided by the square root of the moment of inertia about that radius vector.

If the axes Ox, Oy, Oz are rotated to take new forms Ox', Oy', Oz' , then the equation of the momental ellipsoid (2) takes the form $A'x'^2 + B'y'^2 + C'z'^2 - 2F'y'z' - 2G'z'x' - 2H'x'y' = 1$, where A', B', C', F', G', H' are the moments and products of inertia for the new axes.

Since the coefficients in the equation are of the quadratic change, when the axes rotated, they cannot be scalar quantities like mass. Moreover, these are not the components of a vector. In fact, the array consisting of six coefficients.

$$\begin{bmatrix} A & -H & -G \\ -H & B & -F \\ -G & -F & C \end{bmatrix} \quad \dots(10)$$

is known as *moment of inertia matrix* (or *tensor*) and each quantity is called an element of the matrix (or tensor.) It is also notable that this is a *symmetric matrix* (or tensor.)

Principal axes of inertia. A set of three mutually right angled axes having origin O , fixed in the body and rotating with it such that the products of inertia about them are zero; are known as *principal axes of inertia* or simply *principal axes of the body*.

Let A, B, C, F, G, H be the moments and products of inertia for the old axes Ox, Oy, Oz and A', B', C', F', G', H' , the moments and products of inertia for the new axes Ox', Oy', Oz' which are obtained by rotating the old ones. If we regard Ox', Oy', Oz' as principal axes, then by the definition of principal axes, $F' = G' = H' = 0$ and A', B', C' , are then known as *principal moments of inertia*.

Let (x, y, z) and (x', y', z') be the co-ordinates of a point P with respect to the old and new sets of axes. But origin O being the same in both the sets, the distance of OP and hence OP^2 is invariant in both the sets.

i.e.,
$$x^2 + y^2 + z^2 = x'^2 + y'^2 + z'^2.$$

Also for every point P , the moments of inertia about OP multiplied by OP^2 being invariant, we have

$$Ax^2 + By^2 + Cz^2 - 2Fyz - 2Gzx - 2Hxy = A'x'^2 + B'y'^2 + C'z'^2$$

(since $OP = \frac{1}{\sqrt{I}}$ gives $I \cdot OP^2 = \text{const.}$)

Hence for any arbitrary K , we have the identify

$$Ax^2 + By^2 + Cz^2 - 2Fyz - 2Gzx - 2Hxy - K(x^2 + y^2 + z^2) = A'x'^2 + B'y'^2 + C'z'^2 - K(x'^2 + y'^2 + z'^2) = f \text{ (say).}$$

Consider the equations $\frac{\partial f}{\partial x'} = 0, \frac{\partial f}{\partial y'} = 0, \frac{\partial f}{\partial z'} = 0, \dots(11)$

which give $(A' - K)x' = 0$, $(B' - K)y' = 0$, $(C' - K)z' = 0$,

Neglecting the non-trivial (zero) solution $x' = y' = z' = 0$, these yields

$$K = A', x' = \text{arbitrary}, y' = 0, z' = 0,$$

$$K = B', x' = 0, y' = \text{arbitrary}, z' = 0,$$

$$K = C', x' = 0, y' = 0, z' = \text{arbitrary}.$$

It follows that equations (11) have a non-trivial solution if K is equal to one of the principal moments of inertia A' , B' , C' and the corresponding values of x' , y' , z' , determine the principal axes.

Now, since x , y , z can be expressed in terms of x' , y' , z' as

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial x'} \frac{\partial x'}{\partial x} + \frac{\partial f}{\partial y'} \frac{\partial y'}{\partial x} + \frac{\partial f}{\partial z'} \frac{\partial z'}{\partial x},$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial x'} \frac{\partial x'}{\partial y} + \frac{\partial f}{\partial y'} \frac{\partial y'}{\partial y} + \frac{\partial f}{\partial z'} \frac{\partial z'}{\partial y},$$

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x'} \frac{\partial x'}{\partial z} + \frac{\partial f}{\partial y'} \frac{\partial y'}{\partial z} + \frac{\partial f}{\partial z'} \frac{\partial z'}{\partial z},$$

which give with the help of (11),

$$\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0, \frac{\partial f}{\partial z} = 0,$$

Here K , x' , y' , z' can be so chosen that (11) and (12) hold simultaneously.

The equation (12) gives

$$\left. \begin{aligned} (H - K)x - Hy - Gz &= 0, \\ -Hx - (B - K)y - Fz &= 0, \\ Gx - Fy + (C - K)z &= 0. \end{aligned} \right\} \quad \dots (13)$$

Elimination of x , y , z between these equations by the method of determinant yields

$$\begin{vmatrix} A - K & -H & -G \\ -H & B - K & -F \\ -G & -F & C - K \end{vmatrix} = 0$$

On expansion, it will give a cubic equation in K such that its three roots are the three principal moments of inertia.

It is however, notable in this connection that an axis of symmetry of a rigid body will always be a principal axis.

The momental ellipse. Consider distribution of matter in a plane and fix up Ox and Oy as rectangular axes of reference in this plane. In a plane of symmetry, the normal at O is a principal axis and the section of the momental ellipsoid at O by the plane is a principal section known as *momental ellipse* at O .

Let A , B and H be the moments and products of inertia of a lamina about the axes Ox , Oy in its plane and let

$P(x, y)$ be the particle of mass m distant p from a line ON inclined at an angle θ with Ox . Then

$$p = PN = y \cos \theta - x \sin \theta$$

$$ON = x \cos \theta + y \sin \theta$$

by the co-ordinate geometry.

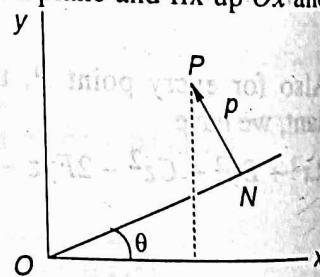


Fig. 15-40

∴ Moment of inertia about ON is given by

$$I = \sum m \cdot p^2 = \sum m (y \cos \theta - x \sin \theta)^2$$

$$= \cos^2 \theta \sum m y^2 + \sin^2 \theta \sum m x^2 - 2 \sin \theta \cos \theta \sum m x y$$

$$= A \cos^2 \theta + B \sin^2 \theta - 2H \sin \theta \cos \theta.$$

Consider a point Q on varying ON , such that $OQ = \frac{1}{\sqrt{I}}$ then

$$A \cos^2 \theta - 2H \sin \theta \cos \theta + B \sin^2 \theta = \frac{1}{OQ^2}.$$

If $OQ = r$, and (x, y) be the co-ordinates of Q , so that $x = r \cos \theta$, $y = r \sin \theta$, then the locus of Q is

$$Ax^2 - 2Hxy + By^2 = 1 \quad \dots(15)$$

which represents an ellipse, known as the *momental ellipse* of the lamina at the point Q .

Again, the products of inertia with respect to the axis ON and a perpendicular axis is

$$\sum m (PN \cdot NO) = \sum m (y \cos \theta - x \sin \theta) (x \cos \theta + y \sin \theta)$$

$$= \sum m xy (\cos^2 \theta - \sin^2 \theta) + \sum m (y^2 - x^2) \sin \theta \cos \theta$$

$$= H \cos 2\theta + \frac{1}{2} (A - B) \sin 2\theta,$$

which vanishes if $\tan 2\theta = \frac{2H}{A - B}$. ∴(16)

This determines the principal axes at O in the plane of the lamina.

Theorem of parallel axes. If the moments and products of inertia of a body about axes through the CG, of a body are given, then moments and products of inertia about all parallel axes can be found.

Let P be a particle of mass m whose coordinates are (x, y, z) with respect to Ox, Oy, Oz and (x', y', z') with respect to $O'x', O'y', O'z'$ where O' is the C.G. of the body and $O'x', O'y', O'z'$ is a set of rectangular axes parallel to the old set Ox, Oy, Oz . If coordinates of O' are $(\bar{x}, \bar{y}, \bar{z})$ with respect to set Ox, Oy, Oz , then

$$x = \bar{x} + x', y = \bar{y} + y', z = \bar{z} + z'.$$

Now the coordinates of C.G. of the system referred to O' as origin are $(0, 0, 0)$

which are also given by $\left(\frac{\sum mx'}{\sum m}, \frac{\sum my'}{\sum m}, \frac{\sum mz'}{\sum m} \right)$ it follows that

$$\sum mx' = 0 = \sum my' = \sum mz'.$$

∴ Moment of inertia of the body about x -axis = $\sum m(y^2 + z^2)$

$$= \sum m \{ (\bar{y} + y')^2 + (\bar{z} + z')^2 \}$$

$$= \sum m (\bar{y}^2 + \bar{z}^2) + \sum m (y'^2 + z'^2) \text{ as } \sum my' = 0 = \sum mz'$$

$$= (\bar{y}^2 + \bar{z}^2) \sum m + \sum m (y'^2 + z'^2)$$

$$= \text{M.I. of the whole mass treated as collected at } G \text{ about } Ox + \text{M.I. of the system about } O'z'.$$

Again, the product of inertia about x, y axes = $\sum mxy$

$$= \sum m (\bar{x} + x') (\bar{y} + y')$$

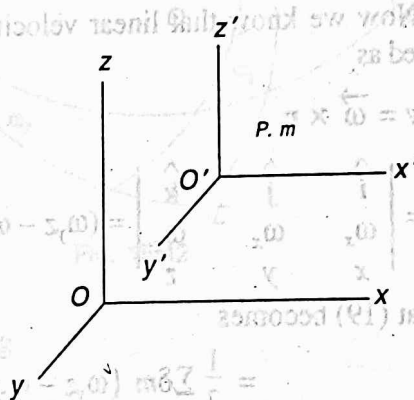


Fig. 15-41

$$= x \bar{y} \sum m + \sum mx' y' \text{ as } \sum my' = 0, \sum mx' = 0 \quad \dots(17)$$

= P.I. of whole mass collected at the C.G. + P.I. of the whole system about parallel axes through C.G.

Theorem of perpendicular axes. Let Ox, Oy, Oz be rectangular axes and let there be a distribution of matter in the plane $z = 0$. If A, B, C denote the moments of inertia about the three axes, we have

$$A = \sum my^2, B = \sum mx^2 \text{ and}$$

$$\text{so that } C = \sum m(x^2 + y^2) = \sum mx^2 + \sum my^2 = B + A \text{ or } C = A + B$$

which gives the theorem of perpendicular axes.

Equimomental systems. The two distributions of matter, having the same total mass and the same principal moments of inertia at the centre of mass (or C.G.) are known as equimomental systems.

Kinetic energy of a rigid body. Suppose that a rigid body is turning about a fixed point O with angular velocity $\vec{\omega}$. Now the kinetic energy of a particle P of mass δm moving with velocity $\mathbf{v}$ is given by $\frac{1}{2} \delta m v^2$, so that the kinetic energy of the whole body is given by

$$T = \frac{1}{2} \sum \delta m \cdot v^2.$$

In order to express T in terms of $\vec{\omega}$ and the principal moments of inertia at O , let us introduce a rectangular set of axes Ox, Oy, Oz with $\hat{i}, \hat{j}, \hat{k}$ unit vectors along them. Then if $\mathbf{r}$ be the position vectors of $P(x, y, z)$ with O origin, we have

$$\mathbf{r} = x\hat{i} + y\hat{j} + z\hat{k}, \quad \dots(19)$$

Now we know that linear velocity of a point $P(\mathbf{r})$ moving with angular velocity $\vec{\omega}$ is defined as

$$\begin{aligned} \mathbf{v} &= \vec{\omega} \times \mathbf{r} \\ &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_x & \omega_y & \omega_z \\ x & y & z \end{vmatrix} = (\omega_y z - \omega_z y)\hat{i} + (\omega_z x - \omega_x z)\hat{j} + (\omega_x y - \omega_y x)\hat{k}. \end{aligned} \quad \dots(20)$$

so that (19) becomes

$$\begin{aligned} &= \frac{1}{2} \sum \delta m \{ (\omega_y z - \omega_z y)^2 + (\omega_z x - \omega_x z)^2 + (\omega_x y - \omega_y x)^2 \} \\ &= \frac{1}{2} [\omega_x^2 \sum \delta m (y^2 + z^2) + \omega_y^2 \sum \delta m (z^2 + x^2) + \omega_z^2 \sum \delta m (x^2 + y^2) \\ &\quad - 2\omega_y \omega_x \sum \delta m yz - 2\omega_z \omega_x \sum \delta m zx - 2\omega_x \omega_y \sum \delta m xy] \end{aligned}$$

$$\text{or } T = \frac{1}{2} [A\omega_x^2 + B\omega_y^2 + C\omega_z^2 - 2F\omega_y\omega_z - 2G\omega_z\omega_x - 2H\omega_x\omega_y], \quad \dots(21)$$

where A, B, C, F, G, H are the moments and products of inertia for x, y, z axes.

In the case when the axes of x, y, z are taken as principal axes of inertia, then

$$F = G = H = 0.$$

so that (21) reduces to

$$T = \frac{1}{2} (A\omega_x^2 + B\omega_y^2 + C\omega_z^2), \quad \dots(22)$$

where A, B, C are principal moments of inertia.

In general, the kinetic energy of a rigid body moving in space is given by

$$T = T_0 + T',$$

where $T_0 = \frac{1}{2} m v_0^2$ = the kinetic energy of the mass-centre

$$\text{and } T' = \frac{1}{2} (A \omega_x^2 + B \omega_y^2 + C \omega_z^2)$$

= the K.E. of motion relative to mass centre,

A, B, C being principal moments of inertia at the mass centre

$$\therefore T = \frac{1}{2} m v_0^2 + \frac{1}{2} (A \omega_x^2 + B \omega_y^2 + C \omega_z^2) \dots (23)$$

The components of angular velocity in terms of Eulerian angles.

To connect the motion of the body in space with the angular velocities of the body about the three moving axes Ox, Oy, Oz , we take O as the centre of a unit sphere and suppose that X, Y, Z and x, y, z are the points in which the sphere is cut by the fixed and moving axes respectively. Let E be the point of intersection of Zz and yx and let $\angle XZz = \phi$, $\angle MZz = \theta$ and $\angle Ezx = \psi$ which are Eulerian angles.

Drop perpendicular zN from z to OZ . Now ψ being the angle made by the plane zOZ with the plane XOZ fixed in space, the velocity of z perpendicular to the plane ZOz is $zN \frac{d\phi}{dt}$ i.e., $\sin \theta \dot{\phi}$ as $zN = Oz \sin \theta$, $Oz = 1$, the radius of unit sphere. Also velocity of z along Zz is $\frac{d\theta}{dt}$

i.e., $\dot{\theta}$. Again the motion of z is expressed by the angular velocities ω_x and ω_y respectively along yz and zx . Hence resolving along and perpendicular to Zz , we get $\dot{\theta} = \omega_x \sin \psi + \omega_y \cos \psi$

and $\sin \theta \dot{\phi} = -\omega_x \cos \psi + \omega_y \sin \psi$ which give on solving

$$\omega_x = \dot{\theta} \sin \psi - \dot{\phi} \sin \theta \cos \psi, \quad \omega_y = \dot{\theta} \cos \psi + \dot{\phi} \sin \theta \sin \psi.$$

Further, by drawing a perpendicular from E on Oz we can show that the velocity of E perpendicular of ZE is $\dot{\phi} \sin ZE = \dot{\phi} \cos \theta$ as $ZE = 90^\circ + \theta$. Also the velocity of x relative to E along Ex is $\dot{\psi} \sin zx = \dot{\psi}$ as $zx = 90^\circ$. Hence the whole velocity of x in space along xy is $\dot{\phi} \cos \theta + \dot{\psi}$ which is the same as angular velocity ω_z along, xy , so that

$$\omega_z = \dot{\phi} \cos \theta + \dot{\psi}.$$

Hence the required relations between θ, ϕ, ψ and $\omega_x, \omega_y, \omega_z$ are

$$\left. \begin{aligned} \omega_x &= \dot{\theta} \sin \psi - \dot{\phi} \sin \theta \cos \psi \\ \omega_y &= \dot{\theta} \cos \psi + \dot{\phi} \sin \theta \sin \psi \\ \omega_z &= \dot{\phi} \cos \theta + \dot{\psi} \end{aligned} \right\} \dots (24)$$

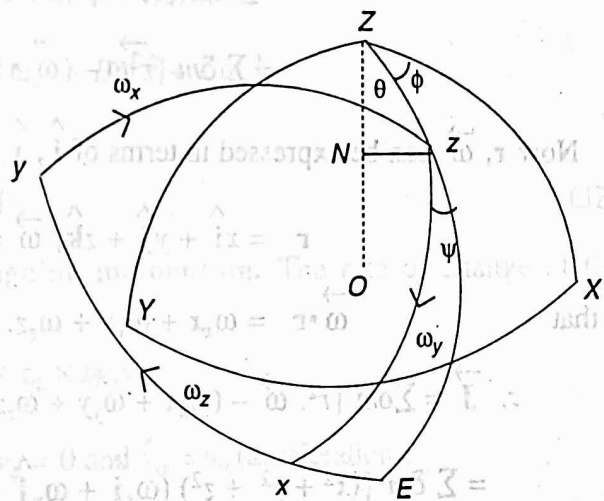


Fig. 15.42

The angular momentum of a rigid body. The angular momentum of a particle of mass m , moving with velocity $\mathbf{v}$ about a fixed point O , is the moment of linear momentum $m\mathbf{v}$ about O ,

i.e.,
$$\vec{\mathbf{J}} = \mathbf{r} \times m\mathbf{v}, \quad \dots(25)$$
 where $\mathbf{r}$ is the position vector of the particle relative to O .

Hence if m_α be the mass of the α th particle and $\mathbf{r}_\alpha$, $\mathbf{v}_\alpha$ denote the position vector relative to O and velocity of α th particle, then the angular momentum of the whole system is given by

$$\vec{\mathbf{J}} = \sum (\mathbf{r}_\alpha \times m_\alpha \mathbf{v}_\alpha) \quad \dots(26)$$

or, in general, considering representative particle of mass δm , position vector $\mathbf{r}$ and velocity $\mathbf{v}$, from the result (26). The angular momentum of a rigid body about a point O is given by

$$\vec{\mathbf{J}} = \sum (\mathbf{r} \times \delta m \mathbf{v}) = \sum \delta m (\mathbf{r} \times \mathbf{v}). \quad \dots(27)$$

But we know that $\mathbf{v} = \vec{\omega} \times \mathbf{r}$ by (20).

$$\begin{aligned} \therefore \vec{\mathbf{J}} &= \sum \delta m (\mathbf{r} \times \vec{\omega} \times \mathbf{r}) \\ &= \sum \delta m \{(\mathbf{r} \cdot \mathbf{r}) \vec{\omega} - (\vec{\omega} \cdot \mathbf{r}) \mathbf{r}\} \\ &= \sum \delta m \{r^2 \vec{\omega} - (\vec{\omega} \cdot \mathbf{r}) \mathbf{r}\}. \quad \dots(28) \end{aligned}$$

Now $\mathbf{r}$, $\vec{\omega}$ can be expressed in terms of $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, $\hat{\mathbf{k}}$ as

$$\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}, \quad \vec{\omega} = \omega_x\hat{\mathbf{i}} + \omega_y\hat{\mathbf{j}} + \omega_z\hat{\mathbf{k}},$$

so that

$$\vec{\omega} \cdot \mathbf{r} = \omega_x x + \omega_y y + \omega_z z.$$

$$\begin{aligned} \therefore \vec{\mathbf{J}} &= \sum \delta m \{r^2 \vec{\omega} - (\omega_x x + \omega_y y + \omega_z z) \mathbf{r}\} \\ &= \sum \delta m \{(x^2 + y^2 + z^2) (\omega_x \hat{\mathbf{i}} + \omega_y \hat{\mathbf{j}} + \omega_z \hat{\mathbf{k}}) - (\omega_x x + \omega_y y + \omega_z z) (x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}})\} \\ &\quad \text{as } r^2 = r^2 = x^2 + y^2 + z^2, \end{aligned}$$

i.e., if J_x, J_y, J_z be the components of $\vec{\mathbf{J}}$,

$$\vec{\mathbf{J}} = J_x \hat{\mathbf{i}} + J_y \hat{\mathbf{j}} + J_z \hat{\mathbf{k}},$$

$$\begin{aligned} \text{so that } J_x \hat{\mathbf{i}} + J_y \hat{\mathbf{j}} + J_z \hat{\mathbf{k}} &= \sum \delta m \{[(x^2 + y^2 + z^2) \omega_x - x(\omega_x x + \omega_y y + \omega_z z)] \hat{\mathbf{i}} + [(x^2 + y^2 + z^2) \omega_y \\ &\quad - y(\omega_x x + \omega_y y + \omega_z z)] \hat{\mathbf{j}} + [(x^2 + y^2 + z^2) \omega_z - z(\omega_x x + \omega_y y + \omega_z z)] \hat{\mathbf{k}}\}. \end{aligned}$$

Comparing the coefficients of $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, $\hat{\mathbf{k}}$ on either side, we get

$$\begin{aligned} J_x &= \sum \delta m \{(x^2 + y^2 + z^2) \omega_x - x(\omega_x x + \omega_y y + \omega_z z)\} \\ &= \omega_x \sum \delta m (y^2 + z^2) - \omega_y \sum \delta m xy - \omega_z \sum \delta m zx \end{aligned}$$

and similarly,

$$\text{i.e. } \left. \begin{aligned} J_x &= A\omega_x - H\omega_y - G\omega_z \\ J_y &= -H\omega_x + B\omega_y - F\omega_z \\ J_z &= -G\omega_x - F\omega_y + C\omega_z \end{aligned} \right\}, \quad \dots(29)$$

where A, B, C, F, G, H are the moments and products of inertia with respect to the triad $\hat{i}, \hat{j}, \hat{k}$.

In case $\hat{i}, \hat{j}, \hat{k}$ are taken as principal axes of inertia at O , then $F = G = H = 0$ and relations, (29), give $J_x = A\omega_x, J_y = B\omega_y, J_z = C\omega_z$, so that

$$\mathbf{J} = A\omega_x \hat{i} + B\omega_y \hat{j} + C\omega_z \hat{k}. \quad \dots(30)$$

In case when the body is constrained to rotate about $\hat{k}$ then $\omega_x = 0, \omega_y = 0$ and $\omega_z = \omega$, so that (29) give $J_x = -G\omega, J_y = -F\omega, J_z = C\omega$, and therefore

$$\vec{J} = -G\omega \hat{i} + F\omega \hat{j} + C\omega \hat{k}. \quad \dots(31)$$

Relation between angular momentum and kinetic energy. We have

$$\vec{\omega} = \omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k}$$

$$\text{and } \mathbf{J} = A\omega_x \hat{i} + B\omega_y \hat{j} + C\omega_z \hat{k}.$$

$$\therefore \vec{\omega} \cdot \mathbf{J} = A\omega_x^2 + B\omega_y^2 + C\omega_z^2.$$

Hence the result (22) can be written as

$$T = \frac{1}{2} \vec{\omega} \cdot \mathbf{J}. \quad \dots(32)$$

The principle of conservation of angular momentum. The rate of change of the angular momentum is given by (26) as

$$\dot{\mathbf{J}} = \sum_{\alpha} (\dot{\mathbf{r}}_{\alpha} \times m_{\alpha} \mathbf{v}_{\alpha} + \mathbf{r}_{\alpha} \times m_{\alpha} \dot{\mathbf{v}}_{\alpha}).$$

But $\dot{\mathbf{r}}_{\alpha} = \mathbf{v}_{\alpha}$, so that $\dot{\mathbf{r}}_{\alpha} \times m_{\alpha} \mathbf{v}_{\alpha} = m_{\alpha} \mathbf{v}_{\alpha} \times \mathbf{v}_{\alpha} = 0$ and $\dot{\mathbf{v}}_{\alpha} = \mathbf{a}_{\alpha}$ (acceleration),

$$\dot{\mathbf{J}} = \sum_{\alpha} (\mathbf{r}_{\alpha} \times m_{\alpha} \mathbf{a}_{\alpha}). \quad \dots(33)$$

Now if $\mathbf{F}_{\alpha}$ and $\mathbf{F}_{\alpha}'$ be the external and internal forces on the α th particle and O is a fixed point, then the acceleration $\mathbf{a}_{\alpha}$ relative to a Newtonian frame of reference or the mass center of the system is given by

$$m_{\alpha} \mathbf{a}_{\alpha} = \mathbf{F}_{\alpha} + \mathbf{F}_{\alpha}'.$$

Thus equation (33) becomes

$$\begin{aligned} \dot{\mathbf{J}} &= \sum_{\alpha} \mathbf{r}_{\alpha} \times (\mathbf{F}_{\alpha} + \mathbf{F}_{\alpha}') \\ &= \sum_{\alpha} \mathbf{r}_{\alpha} \times \mathbf{F}_{\alpha} + \sum_{\alpha} \mathbf{r}_{\alpha} \times \mathbf{F}_{\alpha}'. \end{aligned} \quad \dots(34)$$

Since the internal forces have no moment about any point, therefore $\sum \mathbf{r}_{\alpha} \times \mathbf{F}_{\alpha}' = 0$ and then (34) reduces to

$$\dot{\mathbf{J}} = \sum \mathbf{r}_{\alpha} \times \mathbf{F}_{\alpha} = \vec{\tau},$$

where $\vec{\tau}$ represents total moment of the external forces about the fixed point O .

This gives the principle of conservation of angular momentum, i.e., the rate of change of angular momentum of a rigid body about a fixed point (either fixed or moving

with the mass centre) is equal to the total moment of the external forces about the fixed point.

Further if $\mathbf{a}_0$ be the acceleration of O relative to a Newtonian frame S , then the acceleration of the particle relative to S is $\mathbf{a}_0 + \mathbf{a}_\alpha$ so that the equation of motion of the α th particle is

$$m_\alpha (\mathbf{a}_0 + \mathbf{a}_\alpha) = \mathbf{F}_\alpha + \mathbf{F}'_\alpha.$$

$$\therefore \text{Eqn. (33)} \quad \dot{\mathbf{J}} = \sum_\alpha [\mathbf{r}_\alpha \times (\mathbf{F}_\alpha + \mathbf{F}'_\alpha - m_\alpha \mathbf{a}_0)]$$

$$= \sum_\alpha \mathbf{r}_\alpha \times \mathbf{F}_\alpha + \sum_\alpha \mathbf{r}_\alpha \times \mathbf{F}'_\alpha - \sum_\alpha m_\alpha \mathbf{r}_\alpha \times \mathbf{a}_0$$

Here $\sum \mathbf{r}_\alpha \times \mathbf{F}_\alpha = 0$ as reasoned above and $\sum m_\alpha \mathbf{r}_\alpha = 0$, hence we arrive at the same result as (35)

$$\dot{\mathbf{J}} = \sum_\alpha \mathbf{r}_\alpha \times \mathbf{F}_\alpha = \vec{\tau}.$$

15.15 (g) Moving Frames of Reference. Let S be a Newtonian frame of reference and S' a frame of reference having a motion of translation only relative to S . Also let O be a point in S and O' a point fixed in S' , then a particle A having position vector $\mathbf{r} = \vec{OA}$ and $\mathbf{r}' = \vec{O'A}$, where $\vec{OO'} = \mathbf{r}_0$ is given in positions as

$$\mathbf{r} = \mathbf{r}_0 + \mathbf{r}' \quad \dots(1)$$

Differentiation of equation (1) gives

$$\frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}_0}{dt} + \frac{d\mathbf{r}'}{dt}, \text{ i.e., } \mathbf{v} = \mathbf{v}_0 + \mathbf{v}', \quad \dots(2)$$

where $\mathbf{v}$, $\mathbf{v}'$ are velocities of A relative to S and S' , and $\mathbf{v}_0$ the velocity of S' relative to S .

Further, differentiation of (2) gives $\mathbf{a} = \mathbf{a}_0 + \mathbf{a}'$,

where $\mathbf{a}$, $\mathbf{a}'$, are accelerations of A relative to S and S' , and $\mathbf{a}_0$ the acceleration of S' relative to S .

Now Newton's second law of motion gives for S frame

$$m\mathbf{a} = \mathbf{F} \quad \dots(4)$$

where m is the mass of the particle and $\mathbf{F}$ the force acting on it.

In order to find the motion in S' frame, we substitute the value of $\mathbf{a}$ from equation (3) in (4),

$$m(\mathbf{a}_0 + \mathbf{a}') = \mathbf{F},$$

i.e.

$$m\mathbf{a}' = \mathbf{F} - m\mathbf{a}_0. \quad \dots(5)$$

Here then quantity $-m\mathbf{a}_0$ is termed as *fictitious force*.

Let us now consider a triad of unit orthogonal vectors $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ in a frame of reference S' rotating with angular velocity $\vec{\omega}'$ relative to a Newtonian frame S .

If R_x, R_y, R_z be the components of a vector $\mathbf{R}$, then

$$\mathbf{R} = R_x \hat{\mathbf{i}} + R_y \hat{\mathbf{j}} + R_z \hat{\mathbf{k}}. \quad \dots(6)$$

In order to estimate the rate of change of $\mathbf{R}$ as observed in S , we get differentiating (6),

$$\frac{d\mathbf{R}}{dt} = \frac{dR_x}{dt} \hat{\mathbf{i}} + \frac{dR_y}{dt} \hat{\mathbf{j}} + \frac{dR_z}{dt} \hat{\mathbf{k}} + R_x \frac{d\hat{\mathbf{i}}}{dt} + R_y \frac{d\hat{\mathbf{j}}}{dt} + R_z \frac{d\hat{\mathbf{k}}}{dt} \quad \dots(7)$$

Here $\hat{i}$ being a vector fixed in a rigid body S' rotating with angular velocity $\vec{\omega}'$ may be regarded as the position of a particle B of body S' relative to a base point A , the origin of $\hat{i}$.

Then $\frac{d\hat{i}}{dt}$ gives velocity of B relative to A and hence

$$\text{Similarly, } \left. \begin{aligned} \frac{d\hat{i}}{dt} &= \vec{\omega}' \times \hat{i} \\ \frac{d\hat{j}}{dt} &= \vec{\omega}' \times \hat{j} \\ \frac{d\hat{k}}{dt} &= \vec{\omega}' \times \hat{k} \end{aligned} \right\} \quad (\text{using } \mathbf{v} = \vec{\omega} \times \mathbf{r}). \quad \dots(8)$$

Their substitution in equation (7) yields

$$\begin{aligned} \frac{d\mathbf{R}}{dt} &= \frac{dR_x}{dt} \hat{i} + \frac{dR_y}{dt} \hat{j} + \frac{dR_z}{dt} \hat{k} + \vec{\omega}' \times (R_x \hat{i} + R_y \hat{j} + R_z \hat{k}) \\ &= \frac{D\mathbf{R}}{Dt} + \vec{\omega}' \times \mathbf{R} \end{aligned} \quad \dots(9)$$

$$\frac{D\mathbf{R}}{Dt} \equiv \frac{dR_x}{dt} \hat{i} + \frac{dR_y}{dt} \hat{j} + \frac{dR_z}{dt} \hat{k}. \quad \dots(10)$$

In equation (9) the term $\frac{D\mathbf{R}}{Dt}$ is known as *rate of growth* and $\vec{\omega}' \times \mathbf{R}$ as *rate of transport*.

Thus the rate of change of a vector is the sum of the rate of growth and the rate of transport.

The equation (2) also gives the velocity $\mathbf{v}$ of a moving particle A whose position vector is $\mathbf{r}$, i.e., $\vec{OA} = \mathbf{r}$.

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{D\mathbf{r}}{Dt} + \vec{\omega}' \times \mathbf{r}. \quad \dots(11)$$

Its differentiation gives the acceleration of A ,

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{D^2\mathbf{r}}{Dt^2} = \frac{D\mathbf{v}}{Dt} + \vec{\omega}' \times \mathbf{v}. \quad \dots(12)$$

From equation (12) we find with the help of (11)

$$\mathbf{a} = \frac{D}{Dt} \left(\frac{D\mathbf{r}}{Dt} + \vec{\omega}' \times \mathbf{r} \right) + \vec{\omega}' \times \left(\frac{D\mathbf{r}}{Dt} + \vec{\omega}' \times \mathbf{r} \right) \quad \dots(13)$$

$$\begin{aligned} &= \frac{D^2\mathbf{r}}{Dt^2} + \frac{D\vec{\omega}'}{Dt} \times \mathbf{r} + \vec{\omega}' \times \frac{D\mathbf{r}}{Dt} + \vec{\omega}' \times \frac{D\mathbf{r}}{Dt} + (\vec{\omega}' \times \vec{\omega}') \times \mathbf{r} \\ &= \frac{D^2\mathbf{r}}{Dt^2} + \frac{D\vec{\omega}'}{Dt} \times \mathbf{r} + \vec{\omega}' \times (\vec{\omega}' \times \mathbf{r}) + 2\vec{\omega}' \times \frac{D\mathbf{r}}{Dt}. \end{aligned} \quad \dots(14)$$

$$\text{Denoting } \frac{D\mathbf{r}}{Dt} \text{ by } \mathbf{v}' \text{ and } \frac{D^2\mathbf{r}}{Dt^2} \text{ by } \mathbf{a}' \quad \dots(15)$$

We get

$$\vec{a} = \vec{a}' + \frac{d\vec{\omega}'}{dt} \times \vec{r}' + \vec{\omega}' \times \vec{\omega}' \times \vec{r}' + 2\vec{\omega}' \times \vec{v}$$

Now using

where $\vec{v}'$ and $\vec{a}'$ give the velocity and acceleration of the particle relative to S' , and using

$$\frac{d\vec{\omega}'}{dt} = \frac{D\vec{\omega}}{Dt} + \vec{\omega}' \times \vec{\omega}' \quad \text{on replacing } \vec{R} \text{ by } \vec{\omega}' \text{ in (9),}$$

or

$$\frac{D\vec{\omega}'}{dt} = \frac{d\vec{\omega}'}{dt}$$

$$\begin{aligned} \text{i.e. } \vec{\omega}' \times (\vec{\omega}' \times \vec{r}) &= (\vec{\omega}' \cdot \vec{r}) \vec{\omega}' - (\vec{\omega}' \cdot \vec{\omega}') \vec{r} \\ &= (\vec{\omega}' \cdot \vec{r}) \vec{\omega}' - r \omega'^2 \end{aligned}$$

We get

$$\vec{a} = \vec{a}' + \frac{d\vec{\omega}'}{dt} \times \vec{r} + (\vec{\omega}' \cdot \vec{r}) \vec{\omega}' - r \omega'^2 + 2\vec{\omega}' \times \vec{v}$$

we get from equation (13), $\vec{a} = \vec{a}' + \vec{a}_t + \vec{a}_c$.

...(16)

$$\text{where } \vec{a}_t = \frac{d\vec{\omega}'}{dt} \times \vec{r} + \vec{\omega}' (\vec{\omega}' \cdot \vec{r}) - r \omega'^2, \quad \vec{a}_c = 2\vec{\omega}' \times \vec{v}.$$

...(17)

Here $\vec{a}_t$ is called the *acceleration of transport* and $\vec{a}_c$ the *complementary acceleration of Coriolis*.

The law of motion in S being $m\vec{a} = \vec{F}$ becomes in S' [putting value of $\vec{a}$ from (16)]

$$m\vec{a}' = \vec{F} - m\vec{a}_t - m\vec{a}_c.$$

...(18)

The result (18) may be extended to a general motion of a particle relative to S' as

$$m\vec{a}' = \vec{F} - m\vec{a}_0 - m\vec{a}_t - m\vec{a}_c.$$

...(19)

where $\vec{a}_0$ is the acceleration of the base point O in S' relative to S .

15.15 (h) Euler's Equations of Motion. By equation (30) of § 15.15e, the angular momentum about O , of a rigid body constrained to rotate about the fixed point O is given by

$$\vec{J} = A\omega_x \hat{i} + B\omega_y \hat{j} + C\omega_z \hat{k}.$$

...(1)

and angular velocity $\vec{\omega}'$ of the triad $\hat{i}, \hat{j}, \hat{k}$ as introduced in previous section can be written as

$$\vec{\omega}' = \omega'_x \hat{i} + \omega'_y \hat{j} + \omega'_z \hat{k},$$

where $\omega'_x, \omega'_y, \omega'_z$ are the components of $\vec{\omega}'$.

If the triad is fixed in the body, then $\vec{\omega}' = \vec{\omega}$, the angular velocity of the body.

Hence using (9) of § 15.10 (f), we have

$$\dot{\vec{J}} = \frac{D\vec{J}}{Dt} + \vec{\omega}' \times \vec{J}$$

$$\text{or } \dot{\vec{J}} = A \dot{\omega}_x \hat{i} + B \dot{\omega}_y \hat{j} + C \dot{\omega}_z \hat{k} + (\omega'_x \hat{i} + \omega'_y \hat{j} + \omega'_z \hat{k}) \times (A\omega_x \hat{i} + B\omega_y \hat{j} + C\omega_z \hat{k})$$

$$\begin{aligned} & \text{or } (A\dot{\omega}_x - B\omega_y\omega_z' + C\omega_z\omega_y')\hat{i} + (B\dot{\omega}_y - C\omega_z\omega_x' + A\omega_x\omega_z')\hat{j} \\ & \quad + (C\dot{\omega}_z - A\omega_x\omega_y' + B\omega_y\omega_x')\hat{k} \\ & = \vec{\tau} \text{ since } \vec{J} = \vec{\tau} \\ & = \tau_x\hat{i} + \tau_y\hat{j} + \tau_z\hat{k}, \end{aligned} \quad \dots(2)$$

Comparison of coefficients gives where τ_x, τ_y, τ_z are components of $\vec{\tau}$

$$\left[\begin{aligned} A\dot{\omega}_x - B\omega_y\omega_z' + C\omega_z\omega_y' &= \tau_x, \\ B\dot{\omega}_y - C\omega_z\omega_x' + A\omega_x\omega_z' &= \tau_y, \\ C\dot{\omega}_z - A\omega_x\omega_y' + B\omega_y\omega_x' &= \tau_z. \end{aligned} \right] \quad \dots(3)$$

But when $\hat{i}, \hat{j}, \hat{k}$ are fixed in the body, $\vec{\omega}' = \vec{\omega}$, so that equations (3) become

$$\left[\begin{aligned} A\dot{\omega}_x - (B - C)\omega_y\omega_z &= \tau_x, \\ B\dot{\omega}_y - (C - A)\omega_z\omega_x &= \tau_y, \\ C\dot{\omega}_z - (A - B)\omega_x\omega_y &= \tau_z. \end{aligned} \right] \quad \dots(4)$$

These are known as Euler's equations of motion for a rigid body with a fixed point.

15.16. Poinsot Method

This is a descriptive method giving a good qualitative idea of the motion of a rigid body with a fixed point under no forces. Taking O a fixed point in the body choose a suitable set of mutually perpendicular axes Ox, Oy, Oz . Let A, B, C be the principal moments of inertia O and $\hat{i}, \hat{j}, \hat{k}$ the unit vectors directed along the principal axes at O . The angular velocity $\vec{\omega}$ and the angular momentum $\vec{J}$ are given by

$$\vec{\omega} = \omega_x\hat{i} + \omega_y\hat{j} + \omega_z\hat{k}, \quad \vec{J} = A\omega_x\hat{i} + B\omega_y\hat{j} + C\omega_z\hat{k}. \quad \dots(1)$$

The body under no force means that the forces acting on the body have no moment about O . In view of the fact that the external forces do not work and have no moment about O we may arrive at the conclusion that the kinetic energy T of the body is constant, i.e., by (22) of § 15.14 (f).

$$A\omega_x^2 + B\omega_y^2 + C\omega_z^2 = 2T = \text{constant} \quad \dots(2)$$

and the angular momentum $\vec{J}$ is a vector of constant magnitude and having fixed direction in space so that by (30) of § 15.14 (e), we have

$$J^2 = \vec{J} \cdot \vec{J} = A^2\omega_x^2 + B^2\omega_y^2 + C^2\omega_z^2 = \text{const.}$$

Consider a line OE called as the *invariable line*, through the fixed point O , in the direction of $\vec{J}$ and let another line OQ represent the angular

velocity $\vec{\omega}$ at any time. Draw perpendicular QM from Q to OE , then $OM = \text{scalar projection of } OQ = (\vec{\omega} \cdot \hat{J})$ on the vector $\hat{J} = \text{dot product of } \vec{\omega} \text{ by the unit vector } \hat{J} \text{ (by vector notions)}$

$$\vec{\omega} \cdot \hat{J} = \vec{\omega} \cdot \frac{\vec{J}}{J}$$

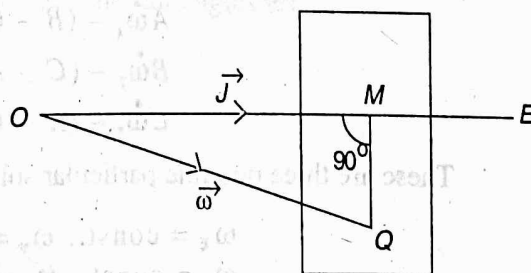


Fig. 15.43

i.e., $\vec{\omega} \cdot \vec{J} = J \cdot OM$.

But by (32) of § 15.15 (e), we have $\vec{\omega} \cdot \vec{J} = 2T$, so that

$$J \cdot OM = 2T$$

which gives

$$OM = \frac{2T}{J} = \text{const.}$$

From (4), we conclude that during the motion the point M remains fixed and therefore the plane through M and perpendicular to the invariable line OE is also fixed and hence it is known as the *invariable plane*. It is also notable that the end point Q of the angular velocity vector $\vec{\omega}$ moves on the invariable plane.

Consider an observer moving with the body, just like a man on the rotating earth. It seems to the observer that the vectors $\hat{i}, \hat{j}, \hat{k}$ are fixed, and $\vec{J}$ and $\vec{\omega}$ both are changing. If the coordinates of Q be (x, y, z) , then it is easy to see that the components $(\omega_x, \omega_y, \omega_z)$ of $\vec{\omega}$ are nothing but (x, y, z) , i.e.,

$$\omega_x = x, \omega_y = y, \omega_z = z.$$

Then (2) and (3) become

$$Ax^2 + By^2 + Cz^2 = 2T.$$

$$A^2x^2 + B^2y^2 + C^2z^2 = J^2$$

which represent two ellipsoids and their intersection is a curve described by Q as observed by the moving observer.

The ellipsoid given (5) is known as **Poinsot ellipsoid** of which the invariable plane is tangent plane at Q .

The tangent plane at $Q (\omega_x, \omega_y, \omega_z)$ is

$$A\omega_x x + B\omega_y y + C\omega_z z = 2T,$$

with direction ratios $A\omega_x, B\omega_y, C\omega_z$, which are the components of angular momentum thereby showing that the normal to the Poinsot ellipsoid is parallel to the line OE and hence the invariable plane is the tangent plane to the Poinsot ellipsoid.

In case the observer S is fixed in space, we get *space cone* on joining the fixed point to the points on the curve described by the angular velocity vector. But if the observer S' is fixed in the body, the angular velocity vector traces out a cone known as *body cone* obtained by joining the fixed point O to the points on the curve.

If we consider the force-free motion of a body, then since the external forces have no moments about O , so that $\tau = 0$ or $\tau_x = \tau_y = \tau_z = 0$, the Euler's equations given by (4) of 15.10 reduce to

$$\left. \begin{aligned} A\dot{\omega}_x - (B - C)\omega_y\omega_z &= 0, \\ B\dot{\omega}_y - (C - A)\omega_z\omega_x &= 0, \\ C\dot{\omega}_z - (A - B)\omega_x\omega_y &= 0, \end{aligned} \right\}$$

These are three possible particular solutions of (8)

$$\left. \begin{aligned} \omega_x &= \text{const.}, \omega_y = 0, \omega_z = 0, \\ \omega_y &= \text{const.}, \omega_z = 0, \omega_x = 0, \\ \omega_z &= \text{const.}, \omega_x = 0, \omega_y = 0. \end{aligned} \right\}$$

The three solutions correspond to steady rotations about these principal axes about which the body spins steadily when the motion is force-free. In view of these solutions and assuming $A > B > C$, the equations (2) and (3) yield.

$$2AT - J^2 > 0, 2CT - J^2 < 0. \quad \dots(9)$$

Writing equations (2) and (3) as

$$A\omega_x^2 + C\omega_z^2 + (B\omega_y^2 - 2T) = 0$$

and

$$A^2\omega_x^2 + C^2\omega_z^2 + (B^2\omega_y^2 - J^2) = 0$$

and solving simultaneously for ω_x^2 and ω_z^2 , we get

$$\frac{\omega_x^2}{C(B^2\omega_y^2 - J^2) - C^2(B\omega_y^2 - 2T)} = \frac{\omega_z^2}{A^2(B\omega_y^2 - 2T) - A^2(B^2\omega_y^2 - J^2)} = \frac{1}{AC^2 - A^2C}$$

$$\text{or } \frac{\omega_x^2}{C(J^2 - 2CT) - BC(B - C)\omega_y^2} = \frac{\omega_z^2}{A(2AT - J^2) - AB(A - B)\omega_y^2} = \frac{AC(A - C)}{AC^2 - A^2C}$$

$$\text{giving } \left\{ \begin{aligned} \omega_x^2 &= \frac{J^2 - 2CT}{A(A - C)} - \frac{B(B - C)}{A(A - C)}\omega_y^2, \\ \omega_z^2 &= \frac{2AT - J^2}{C(A - C)} - \frac{B(A - B)}{C(A - C)}\omega_y^2. \end{aligned} \right\} \dots(10)$$

Substituting these values of ω_x^2 and ω_z^2 in second equation of (8), i.e.,

$$\begin{aligned} \omega_y^2 &= \left(\frac{C - A}{B} \right)^2 \omega_x^2 \omega_z^2 \\ &= \left(\frac{C - A}{B} \right)^2 \left\{ \frac{J^2 - 2CT}{A(A - C)} - \frac{B(B - C)}{A(A - C)}\omega_y^2 \right\} \times \left\{ \frac{2AT - J^2}{C(A - C)} - \frac{B(A - B)}{C(A - C)}\omega_y^2 \right\} \dots(11) \end{aligned}$$

Its solution will give ω_y and hence ω_x and ω_z by (10).

15.18. Motion of a Symmetric Top in Gravitational Field Using Lagrangian Technique

Symmetric Top in Gravitational Field using Lagrangian Technique

A symmetric top is defined as a material body having symmetry about an axis and terminating in a sharp point at one end of the axis, called the apex or the vertex.

We shall now describe the motion of such a top in a gravitational field.

When the motion of the top is symmetrical about z-axis, we have $A = B$ and then the kinetic energy T in terms of angular velocity $\vec{\omega}$, principal moments A and C , is given by

$$T = \frac{1}{2} A (\omega_x^2 + \omega_y^2) + \frac{1}{2} C \omega_z^2,$$

while angular velocity components, $\omega_x, \omega_y, \omega_z$ in terms of Eulerian angles are

$$\omega_x = \dot{\phi} \sin \theta \sin \psi + \dot{\theta} \cos \psi,$$

$$\omega_y = \dot{\phi} \sin \theta \cos \psi - \dot{\theta} \sin \psi,$$

$$\omega_z = \dot{\phi} \cos \theta + \dot{\psi}.$$

so that $\omega_x^2 + \omega_y^2 = \dot{\phi}^2 \sin^2 \theta + \dot{\theta}^2$.

$$\begin{aligned} \therefore T &= \frac{1}{2} A (\dot{\phi}^2 \sin^2 \theta + \dot{\theta}^2) \\ &\quad + \frac{1}{2} C (\dot{\phi} \cos \theta + \dot{\psi})^2 \end{aligned} \dots(1)$$

and the potential energy V of the top whose centre of mass K is at distance a from the origin O is given by

$$V = mgl \cos \theta. \quad \dots(2)$$

Lagrangian L is given by

$$L = T - V = \frac{1}{2} A \dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta + \frac{1}{2} C (\dot{\psi} + \dot{\phi} \cos \theta)^2 - mgl \cos \theta. \quad \dots(3)$$

The total energy E is given by

$$E = T + V = \frac{1}{2} A (\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta) + \frac{1}{2} C (\dot{\psi} + \dot{\phi} \cos \theta)^2 + mgl \cos \theta. \quad \dots(4)$$

The component of the angular momentum for ϕ is vertical and for ψ is along z -axis in the body. Also the torque of gravity being along the line of nodes has no component either along the vertical or in the direction of z -axis as both of these axes are perpendicular to the line of nodes. As such the components of angular momentum along these two axes must be constant.

$$\text{i.e., } J_{\psi} = \frac{\partial L}{\partial \dot{\psi}} = C (\dot{\psi} + \dot{\phi} \cos \theta) = C \omega_z = \text{constant } Cs = \mu \quad \dots(5)$$

and

$$\begin{aligned} J_{\phi} &= \frac{\partial L}{\partial \dot{\phi}} \\ &= A \dot{\phi} \sin^2 \theta + C (\dot{\psi} + \dot{\phi} \cos \theta) \cos \theta \\ &= A \dot{\phi} \sin^2 \theta + Cs \cos \theta \text{ by equation (5)} \\ &= \text{constant} = \lambda \text{ (say),} \end{aligned}$$

Thus

$$A \dot{\phi} \sin^2 \theta + Cs \cos \theta = \lambda. \quad \dots(6)$$

Putting these values in (4), we get

$$2(E - mga \cos \theta) = A (\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta) + Cs^2$$

or

$$A (\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta) + \frac{\mu^2}{C} = 2(E - mga \cos \theta) \quad \dots(7)$$

and from equation (6),

$$A \dot{\phi} \sin^2 \theta = \lambda - \mu \cos \theta \quad \dots(8)$$

Ex. 54. For the motion of a top, if $\mu = \cos \theta$, prove that the equations of motion can be put in the form

$$\dot{u}^2 = \frac{1}{A} \left(2E - \frac{\mu^2}{C} - 2mga u \right) (1 - u^2) - \frac{(\lambda - \mu u)^2}{A^2} \equiv f(u), \text{ where } \dot{u}^2 = f(u),$$

and

$$\dot{\phi} = \frac{\lambda - \mu u}{A (1 - u^2)}.$$

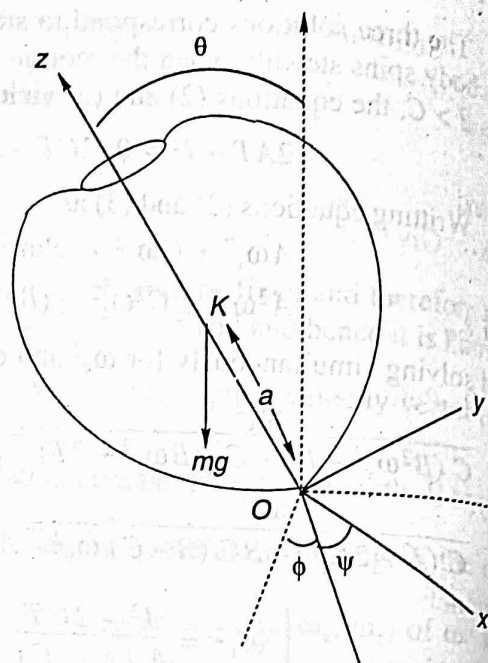


Fig. 15-48

Solution. The equations of motion of a top as derived in section 15.18 are

$$A(\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta) + \frac{\mu^2}{C} = 2(E - mga \cos \theta), \quad \dots(1)$$

$$\text{and} \quad A\dot{\phi} \sin^2 \theta = \lambda - \mu \cos \theta. \quad \dots(2)$$

Given that $u = \cos \theta$.

$$\text{i.e.,} \quad \dot{u} = -\sin \theta \dot{\theta} \quad \text{or} \quad \dot{\theta} = -\frac{\dot{u}}{\sin \theta} = -\frac{\dot{u}}{\sqrt{1-u^2}}$$

Putting $\cos \theta = u$, i.e., $\sin^2 \theta = 1 - \cos^2 \theta = 1 - u^2$ in (2), we get

$$A\dot{\phi}(1-u^2) = \lambda - \mu u$$

$$\dot{\phi} = \frac{\lambda - \mu u}{A(1-u^2)}$$

which is one of the required relations.

Again putting this value of $\dot{\phi}$ and $u = \cos \theta$ in (1), we find

$$A \left[\frac{\dot{u}^2}{1-u^2} + \frac{(\lambda - \mu u)^2}{A^2(1-u^2)^2} (1-u^2) \right] + \frac{\mu^2}{C} = 2(E - mga u)$$

$$\text{or } f(u) \equiv \dot{u}^2 = \frac{1}{A} \left\{ 2E - \frac{\mu^2}{C} - 2mga u \right\} (1-u^2) - \frac{(\lambda - \mu u)^2}{A^2} \quad \dots(4)$$

$$\text{where } \dot{u}^2 = f(u). \quad \dots(5)$$

Ex. 55. Discuss the motion of the top, when it is spinning with its axis fixed in position and then released.

Solution. In Ex. 59, we have proved that

$$f(u) \equiv \dot{u}^2 = \frac{1}{A} \left\{ 2E - \frac{\mu^2}{C} - 2mga u \right\} (1-u^2) - \frac{(\lambda - \mu u)^2}{A^2}. \quad \dots(1)$$

Let us impose the initial conditions as

$$u = u_0, \text{ when } \dot{u} = 0 \text{ and } \dot{\phi} = 0,$$

so that the relation $\dot{\phi} = \frac{\lambda - \mu u}{A(1-u^2)}$ derived in Ex. 59, gives $\lambda = \mu u_0$ and then (1) gives

$$0 = \frac{1}{A} \left\{ 2E - \frac{\mu^2}{C} - 2mga u_0 \right\} (1-u_0^2).$$

$$\text{i.e.,} \quad E = \frac{1}{2} \frac{\mu^2}{C} + mga u_0.$$

Putting these values of E and λ in (1), we get

$$f(u) = \frac{2mga}{A} (u_0 - u) (1-u^2) - \frac{\mu^2}{A^2} (u_0 - u)^2 \quad \dots(2)$$

$$= (u_0 - u) \left\{ \frac{2mga}{A} (1-u^2) - \frac{\mu^2}{A^2} (u_0 - u) \right\},$$

which follows that one root of $f(u) = 0$ is u_0 .

If we draw a graph of $f(u)$ then it is found like that given in the adjoining diagram. Here u_1, u_2, u_3 are the roots of the cubic (1).

Differentiation of equation (2) gives

$$\begin{aligned} f'(u) &= -\frac{2mga}{A}(1-u^2) \\ &= -2u \times \frac{2mga}{A}(u_0-u) \\ &\quad + \frac{2\mu}{A^2}(u_0-u), \end{aligned}$$

$$\text{so that } f'(u_0) = -\frac{2mga}{A}(1-u_0^2)$$

which is negative, showing that the root μ_2 of (2) belongs to μ_0 , i.e. $u_2 = u_0$. As such, it is clear that the oscillation of u is from u_1 to $u_3 (= u_3)$, u_1 being the smallest root of $f(u)$.

In order to find the other roots of the cubic (2), let us make a substitution,

$$v = \frac{\mu^2}{4A m g a} = \frac{C^2 s^2}{4A m g a} \quad \dots(3)$$

Then $f(u) = 0$, from (2), gives

$$\frac{2mga}{A}(u_0-u)(1-u^2) - \frac{4v m g a}{A}(u_0-u)^2 = 0.$$

$$\text{or } 1-u^2-2v(u_0-u) = 0$$

$$\text{or } u^2-2vu-(1-2vu_0) = 0$$

$$u = \frac{2v \pm \sqrt{4v^2 + 4(1-2vu_0)}}{2}$$

$$= v \pm \sqrt{(v^2 - 2vu_0 + 1)}.$$

Hence

$$u_1 = v - \sqrt{(v^2 - 2vu_0 + 1)},$$

$$u_2 = u_0,$$

$$u_3 = v + \sqrt{(v^2 - 2vu_0 + 1)}$$

Suppose that $\theta_0, \theta_1, \theta_2$ are values of θ corresponding to the three values u_1, u_2, u_3 of u according to the relation $u = \cos \theta$. Then the axis of the top falls from an inclination θ_0 to θ_1 .

$$\text{where } \cos \theta_1 = u_1 = v - \sqrt{(v^2 - 2vu_0 + 1)}. \quad \dots(5)$$

It then starts to rise again and swings up to $\theta_1 = \theta_0$, where the axis is instantaneously at rest. When v is large, then by the relation (3), the spin is also large.

Now from eqn. (5), we have

$$\begin{aligned} \cos \theta_1 &= v - v \left\{ 1 + \left(\frac{1}{v^2} - \frac{2u_0}{v} \right) \right\}^{1/2}, \text{ } v \text{ being large, } \frac{1}{v} \text{ is small} \\ &= v - v \left\{ 1 - \left(\frac{2u_0}{v} - \frac{1}{v^2} \right) \right\}^{1/2} \end{aligned}$$

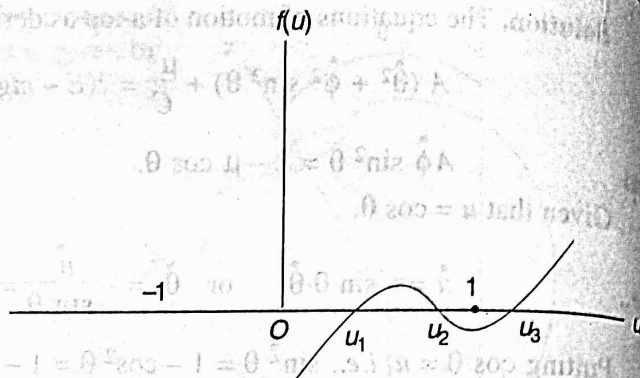


Fig. 15-49

$$\begin{aligned}
 &= v - v \left[1 - \frac{1}{2} \left(\frac{2u_0}{v} - \frac{1}{v^2} \right) \right]^{1/2} \\
 &= v - v \left[1 - \frac{1}{2} \left(\frac{2u_0}{v} - \frac{1}{v^2} \right) \right] - \frac{1}{8} \left(\frac{2u_0}{v} \right)^2 \quad \text{neglecting higher order terms} \\
 &= v - v + u_0 - \frac{1}{2v} - \frac{1}{2} \frac{u_0^2}{v} \\
 &= u_0 - \frac{1}{2v} (1 - u_0^2) \\
 &= \cos \theta_0 - \frac{1}{2v} (1 - \cos^2 \theta_0) \\
 &= \cos \theta_0 - \frac{\sin^2 \theta_0}{2v}
 \end{aligned}$$

$$\begin{aligned}
 \text{or } \cos \theta_1 - \cos \theta_0 &= -\frac{\sin^2 \theta_0}{2v} \\
 &= -(\theta_1 - \theta_0) \sin \theta_0.
 \end{aligned}$$

The difference $\theta_1 - \theta_0$ being small, this approximation can be used, where

$$\theta_1 - \theta_0 = \frac{1}{2v} \sin \theta_0 = \frac{2A m g a}{C^2 s^2} \sin \theta_0. \quad \text{by (3)}$$

Now the differential equation of the path in a unit sphere is

$$\frac{d\phi}{du} = \frac{\dot{\phi}}{u} \text{ where } \dot{\phi} = \frac{\lambda - \mu u}{A(1-u^2)} = \frac{u(u_0 - u)}{A(1-u^2)}, \text{ when } \lambda = \mu u_0,$$

$$\text{and } \dot{u} = \pm \sqrt{f(u)}.$$

$$\text{so that } \frac{d\phi}{du} = \frac{\mu(\mu_0 - u)}{\pm A(1-u^2)\sqrt{f(u)}}$$

When $u \rightarrow u_0$, $f(u) \rightarrow 0$ and $\frac{d\phi}{du} = 0$, showing that the path meets the circle $\theta = \theta_0$ at right angles and the path has cusps at these points. Hence the given example is an example of the *cuspidal motion of the top*.

15.19. Condition for the Stability of a Sleeping Top

Solution. When the axis remains stationary and there is no apparent motion of the top as a whole, then it is said to be a *sleeping top*.

The cubic derived in Section 15.18 for the motion of a top is

$$f(u) = \frac{1}{A} \left[2E - \frac{\mu^2}{C} - 2mga u \right] (1-u^2) - \frac{(\lambda - \mu u)^2}{A^2} \quad \dots(1)$$

where $\mu = Cs$ and $\lambda = A\dot{\phi} \sin^2 \theta + Cs \cos \theta$.

For a sleeping top, we have $\theta = 0 = \dot{\theta}$, so that

$$A\dot{\phi} \sin^2 \theta + Cs \cos \theta = \lambda, \text{ given } \lambda = \mu = Cs$$

and the equation of motion for the top

A $(\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta) + \frac{\mu^2}{C} = 2(E - mga \cos \theta)$ gives $E = \frac{1}{2} Cs^2 + mga$.

Substituting these values of λ , and E in (1), we get

$$\begin{aligned} f(u) &= \frac{1}{A} [Cs^2 + 2mga - Cs^2 - 2mga u] (1 - u^2) - \frac{C^2 s^2}{A^2} (1 - u)^2 \\ &= \frac{1}{A} \cdot 2mga (1 - u) (1 - u^2) - \frac{C^2 s^2}{A^2} (1 - u)^2 \\ &= \frac{1}{A} \cdot 2mga (1 - u^2) (1 - u) - \frac{C^2 s^2}{A^2} (1 - u)^2 \\ &= (1 - u)^2 \left[\frac{2mga}{A} (1 + u) - \frac{C^2 s^2}{A^2} \right] \end{aligned} \quad \dots(2)$$

$$\therefore f'(u) = -2(1 - u) \left[\frac{2mga}{A} (1 + u) - \frac{C^2 s^2}{A^2} \right] + (1 - u)^2 \cdot \frac{2mga}{A}$$

For maxima or minima of u , $f'(u) = 0$, gives

$$(1 - u) \left[-2 \left\{ \frac{2mga}{A} (1 + u) - \frac{C^2 s^2}{A^2} \right\} + (1 - u) \frac{mga}{A} \right] = 0.$$

The first factor gives $u = 1$ and the second one gives

$$\frac{2mga}{A} \{1 - u - 2(1 + u)\} + \frac{2C^2 s^2}{A^2} = 0$$

or $-mga(1 + 3u) + \frac{C^2 s^2}{A} = 0$, i.e., $u = \frac{1}{3} \left(\frac{C^2 s^2}{Amga} - 1 \right).$

Now

$$\begin{aligned} f''(u) &= 2 \left[\frac{2mga}{A} (1 + u) - \frac{C^2 s^2}{A^2} \right] - 2(1 - u) \cdot \frac{2mga}{A} - 2(1 - u) \cdot \frac{2mga}{A} \\ &= 2 \left[\frac{2mga}{A} - \frac{C^2 s^2}{A^2} \right] \text{ for } u = 1. \end{aligned}$$

This will be -ve if $\frac{C^2 s^2}{A^2} > \frac{4mga}{A}$, i.e., if $s^2 > \frac{4Amga}{C^2}$ and then the motion will be stable.

Hence for the stability of a sleeping top, the condition is

$$s^2 > \frac{4Amga}{C^2}$$

In the limiting equilibrium $s^2 = \frac{4Amga}{C^2}$ which renders the other root $u = \frac{1}{2} \left(\frac{C^2 s^2}{4mga} - 1 \right)$ to be unity. Hence the second root gives the position of limiting equilibrium.

15.20. Non-linear Dynamics and Chaos

(a) Introduction

A system is said to be linear if the time-evolution equations are linear.

For example consider a mass spring-system as shown in Fig. The length of spring is along X-axis, mass m is connected to one end of a spring whose other end is tied to a rigid support. A force F is applied to stretch the spring. If force is small, so that spring obeys

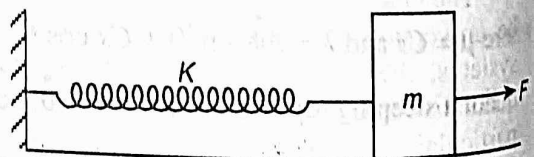


Fig. 15-50

Hooke's law and x is the instantaneous extension produced in the spring, then restoring force in the spring is

$$F = -kx$$

where k is spring constant, which is a measure of stiffness of spring.

According to the Newton's second law

$$F = m \frac{d^2x}{dt^2}$$

so

$$m \frac{d^2x}{dt^2} = -kx$$

or

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x \quad \dots(1)$$

This is differential equation of simple harmonic motion. It means that if the spring is stretched and released, the mass executes simple harmonic motion about equilibrium position with angular frequency

$$\omega = \sqrt{\frac{k}{m}}$$

Equation (1) is linear in x and acceleration $\frac{d^2x}{dt^2}$; therefore the system is linear.

On the other hand if the spring loses its elastic behaviour; so that it does not obey Hooke's law, then the force is no longer proportional to displacement, it may happen that

$$F = bx^2$$

so that

$$m \frac{d^2x}{dt^2} = bx^2$$

or

$$\frac{d^2x}{dt^2} = \frac{b}{m}x^2$$

This equation is non-linear, so the system is said to be non-linear.

Physically if we say force as action and displacement x as response, then the system is linear if the action is proportional to response. For example, suppose system is given a 'kick' and certain response is observed. If now the kick is made twice hard; if the response is doubled, the system is said to be *linear*. On the other hand if the response is not doubled (it may be larger or smaller), then the system is said to be *non-linear*. Let $u(x, t)$ be the response of system to a certain action $F(t)$. If we change $F(t)$ to $2F(t)$, then a linear system will have a response $2u(x, t)$. For a non-linear system the response will be larger or smaller than $2u(x, t)$. The study of non-linear behaviour of the system is called the *non-linear dynamics*.

A system is said to be linear if and only if it satisfies the following condition :

If $u(x, t)$ and $v(x, t)$ are linearly independent solutions of time evolution equation for the system, then linear combination of u and v , i.e., $\alpha u(x, t) + \beta v(x, t)$ is also a solution, where α and β are numbers.

The term '*Chaos*' is used to describe the apparently complex behaviour of systems which we consider to be simple and well-behaved.

The chaotic behaviour of system is due to non-linearity. The chaotic behaviour appears to be universal. It is observed in mechanical oscillators, electrical circuits, non-linear optical systems, chemical reactions, heated fluids, lasers, etc., the chaotic behaviour shows dramatic qualitative and quantitative universal features. These features are independent of the details of particular system. The universality of chaotic behaviour means that we learn the chaotic behaviour by studying mechanical system; it may be applied to electrical systems and heated fluids.

15.20. (b) Non-linear Motion in one Dimension

Consider the motion of a particle of mass m along X-axis in conservative force-field given by

$$F(x) = -\frac{dV(x)}{dx}$$

where $V(x)$ is potential energy.

The Lagrangian of system is

$$L = \frac{1}{2} m \dot{x}^2 - V(x) \quad \dots(1)$$

In conservative system, the total energy

$$E = \frac{1}{2} m \dot{x}^2 + V(x) \text{ is constant of motion.}$$

The potential function $V(x)$ may be expanded by Taylor's series about the equilibrium position $x = x_0$, so that

$$V(x) = V(x_0) + \left(\frac{dV}{dx}\right)_{x=x_0} (x - x_0) + \left(\frac{d^2V}{dx^2}\right)_{x=x_0} \frac{(x - x_0)^2}{2!} + \left(\frac{d^3V}{dx^3}\right)_{x=x_0} \frac{(x - x_0)^3}{3!} + \dots \quad (2)$$

Shifting the origin such that potential energy V is zero at $x = x_0$.

(In stable equilibrium position x_0 , $V(x_0) = \text{minimum}$, which may be taken as zero without any loss in generality)

$$\therefore V(x) = \left(\frac{dV}{dx}\right)_{x=x_0} (x - x_0) + \left(\frac{d^2V}{dx^2}\right)_{x=x_0} \frac{(x - x_0)^2}{2!} + \left(\frac{d^3V}{dx^3}\right)_{x=x_0} \frac{(x - x_0)^3}{3!} + \dots$$

At equilibrium V at x_0 is minimum, so $\left(\frac{dV}{dx}\right)_{x=x_0}$ is zero.

$$\therefore V(x) = \left(\frac{d^2V}{dx^2}\right)_{x=x_0} \frac{(x - x_0)^2}{2!} + \left(\frac{d^3V}{dx^3}\right)_{x=x_0} \frac{(x - x_0)^3}{3!} + \dots$$

In this limit of Small Oscillations

$$V(x) = \left(\frac{d^2V}{dx^2}\right)_{x=x_0} \frac{(x - x_0)^2}{2}$$

Shifting the origin at $x = x_0$, we may put displacement $x - x_0 = X$, so that

$$V(x) = \frac{1}{2} k X^2$$

where $k = \left(\frac{d^2V}{dx^2}\right)_{x=x_0}$ and force field $F(x) = -\frac{dV}{dx} = -kX$ for symmetrical case about $x = x_0$

This is linear system.

However, for a non-linear system, the first correction term to a linear form is proportional to x^3 , i.e.,

$$F(x) = -kx + \epsilon x^3$$

where $\epsilon = \frac{1}{3!} \left(\frac{\partial^4 V}{\partial x^4}\right)_{x=x_0}$ is a small quantity.

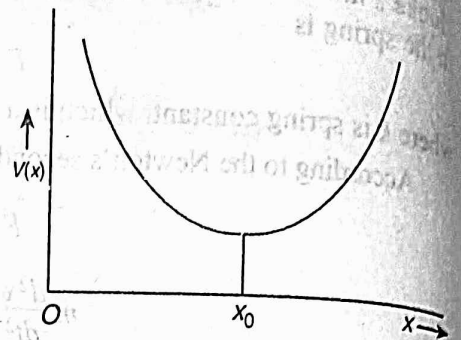


Fig. 15-51

The corresponding potential function is

$$V(x) = \frac{1}{2} kx^2 - \frac{1}{4} \epsilon x^4$$

For $\epsilon > 0$, $F(x)$ is less than the linear term alone and the system is said to be *soft* while for $\epsilon < 0$, $F(x)$ is greater than the linear term alone, so the system is said to be *hard*.

In simplest case of *asymmetric non-linear force*

$$F(x) = -kx + \lambda x^2$$

and the potential energy $u(x) = \frac{1}{2} kx^2 - \frac{1}{3} \lambda x^3$.

For $x > 0$, the system is hard and for $x < 0$, the system is soft.

15.20. (c) Pendulum Equation and Solution

Consider a pendulum having a bob of mass ' m ' and a massless inextensible rod of length l , free to oscillate under influence of gravity. Let pendulum be displaced through an angle θ_0 from equilibrium position SO and released to oscillate freely. If θ is angular displacement at any time from equilibrium position, then assuming reference level for zero potential energy as a horizontal line passing through equilibrium position O of bob;

Potential energy

$$\begin{aligned} V &= mgh = mg(l - l \cos \theta) \\ &= mgl(1 - \cos \theta) \end{aligned}$$

Kinetic energy of bob, $T = \frac{1}{2} m \dot{x}^2$

But $\dot{x} = l\dot{\theta}$

$$\therefore T = \frac{1}{2} ml^2 \dot{\theta}^2$$

Total energy of system $E = T + V$

$$= \frac{1}{2} ml^2 \dot{\theta}^2 + mgl(1 - \cos \theta) = \text{constant.}$$

To determine the constant E , we use initial condition that at $\theta = \theta_0$, angular velocity $\dot{\theta} = 0$.

$$\therefore E = mgl(1 - \cos \theta_0) \quad \dots(2)$$

Substituting this value in equation (1), we get

$$mgl(1 - \cos \theta_0) = \frac{1}{2} ml^2 \dot{\theta}^2 + mgl(1 - \cos \theta)$$

$$\Rightarrow \dot{\theta}^2 = \frac{2g}{l} (\cos \theta - \cos \theta_0) \quad \dots(3)$$

To obtain equation of pendulum, we differentiate equation (3) with respect to time t

$$2\dot{\theta}\ddot{\theta} = -\frac{2g}{l} \sin \theta \dot{\theta}$$

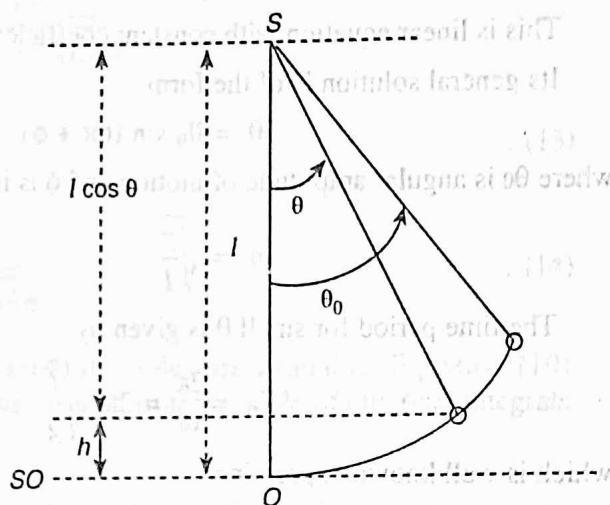


Fig. 15-52

$$\Rightarrow \ddot{\theta} + \frac{g}{l} \sin \theta = 0 \quad \dots(4)$$

This is *equation of motion of pendulum*. This is non-linear differential equation due to presence of trigonometric term $\sin \theta$.

Maclaurin series expansion of $\sin \theta$ is

$$\sin \theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} \dots \dots \dots \quad \dots(5)$$

Under small amplitude approximation $\sin \theta \approx \theta$, so that equation (4) takes the form

$$\ddot{\theta} + \frac{g}{l} \theta = 0 \text{ or } \frac{d^2\theta}{dt^2} + \omega^2 \theta = 0$$

where

$$\omega^2 = g/l$$

This is linear equation with constant coefficients

Its general solution is of the form

$$\theta = \theta_0 \sin(\omega t + \phi)$$

where θ_0 is angular amplitude of motion and ϕ is initial phase, with angular frequency

$$\omega = \sqrt{\frac{g}{l}}$$

The time period for small θ is given by

$$T_\theta = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{l}{g}} \quad \dots(7)$$

which is well known expression.

In case when amplitude of oscillations is large, we have from equation (3)

$$\dot{\theta} \left(= \frac{d\theta}{dt} \right) = \sqrt{\frac{2g}{l}} (\cos \theta - \cos \theta_0)$$

$$\Rightarrow dt = \sqrt{\frac{l}{2g}} \frac{d\theta}{\sqrt{\cos \theta - \cos \theta_0}}$$

The time period of oscillations is twice the time taken by the pendulum to swing from $\theta = -\theta_0$ to $\theta = +\theta_0$, so that

$$T_l = 2 \sqrt{\frac{l}{2g}} \int_{-\theta_0}^{\theta_0} \frac{d\theta}{\sqrt{\cos \theta - \cos \theta_0}} \quad \dots(8)$$

Using $\cos \theta = 1 - 2 \sin^2 \frac{\theta}{2}$ and $\cos \theta_0 = 1 - 2 \sin^2 \frac{\theta_0}{2}$, we get

$$T_l = \sqrt{\frac{l}{g}} \int_{-\theta_0}^{\theta_0} \frac{d\theta}{\sqrt{\sin^2 \frac{\theta_0}{2} - \sin^2 \frac{\theta}{2}}} \quad \dots(9)$$

Introducing two new variables K and ϕ such that

$$K = \sin \frac{\theta_0}{2} \quad \dots(10)$$

$$\text{and } \sin \frac{\theta}{2} = k \sin \phi \quad \dots(11)$$

Differentiating (11)

$$\frac{1}{2} \cos \left(\frac{\theta}{2} \right) d\theta = k \cos \phi d\phi$$

$$\Rightarrow \frac{d\theta}{\cos \left(\frac{\theta}{2} \right)} = \frac{2k \cos \phi d\phi}{\sqrt{1 - \sin^2 \frac{\theta}{2}}} \quad \dots(12)$$

$$\Rightarrow \text{or } d\theta = \frac{2k \cos \phi d\phi}{\sqrt{1 - k^2 \sin^2 \phi}}$$

Now equation (9) becomes

$$T_1 = 4 \sqrt{\frac{l}{g}} \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}} \quad \dots(13)$$

$$= 4 \sqrt{\frac{l}{g}} K(k)$$

where

$$K(k) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}} \quad \dots(14)$$

represents complete elliptical integral of first kind. Its tables are available. Equation (10) determines k in terms of maximum angle of swing θ_0 , then from the table of elliptical integrals we may determine $K(k)$.

For $\theta_0 = 20^\circ$, we have

$$(T)_{20^\circ} = 1.000076 T_0$$

When

 $\theta_0 = 60^\circ$, we have

$$(T)_{60^\circ} = 1.07 T_0$$

Clearly the period of pendulum depends upon the amplitude of oscillation. Larger is the amplitude of oscillations, larger is the period.

15.20. (d) Phase Plane Analysis of Dynamical Systems : Autonomous Systems

If the nonlinear differential equation under investigation is of second order and does not contain the independent variable time t explicitly, then it is possible to extract some information regarding the properties of the solution by a geometrical method without actually going into tedious solution of the equation. The dynamical systems whose equations of motion do not involve time t explicitly are called 'autonomous systems'. The equations of motion of second order autonomous systems have the general form

$$\ddot{x} + g(\dot{x}) + f(x) = 0 \quad \dots(1)$$

If the velocity $(\dot{x})$ of system is treated as other independent variable and denoted by $y (= \dot{x})$, then equation (1) may be reduced to two first order differential equations as

$$\left. \begin{aligned} \frac{dx}{dt} &= y & \dots(a) \\ \frac{dy}{dt} &= -[g(y) + f(x)] & \dots(b) \end{aligned} \right\} \quad \dots(2)$$

and

Equations (2) represent special case of more general system

$$\left. \begin{aligned} \frac{dx}{dt} &= P(x, y) & \dots(a) \\ \text{and} \quad \frac{dy}{dt} &= Q(x, y) & \dots(b) \end{aligned} \right\} \dots(3)$$

where $P(x, y)$ and $Q(x, y)$ are the functions of variables x and y .

As already pointed out that the phase space is the superposition of position and momentum (or velocity) space and in phase space the position and velocity coordinates are treated on equal footings.

So (x, y) is a point in *phase plane*. Dividing equation (2b) by (2a), we get

$$\frac{dy}{dx} = -\frac{g(y) + f(x)}{y}, \quad y \neq 0 \quad \dots(4)$$

This differential equation defines a definite curve in the phase space. This curve is called the *phase trajectory* or simply the *trajectory* of the system in the phase space. The phase trajectory of system defined by general equations (3) is

$$\frac{dy}{dx} = \frac{Q(x, y)}{P(x, y)} \quad \dots(5)$$

Singular Point. A point (x_0^*, y_0^*) of the phase plane for which $P(x_0^*, y_0^*) = Q(x_0^*, y_0^*) = 0$ is called a *singular point*. All the points of phase plane for which the functions $P(x, y)$ and $Q(x, y)$ do not vanish are called *ordinary points*. For time derivative of x and y coordinates are taken as the x and y components of velocity of system, i.e.,

$$v_x = \frac{dx}{dt} = P(x, y); \quad v_y = \frac{dy}{dt} = Q(x, y) \quad \dots(6)$$

The ordinary point (x, y) may be taken as a representative point of the system. At a singular point (x_0^*, y_0^*) , we have

$$v_x = 0 \quad \text{and} \quad v_y = 0 \quad \dots(7)$$

Thus a singular point may be regarded as the position of equilibrium with zero velocity.

Illustrations of Phase Trajectories

1. Linear Harmonic Oscillator

The differential equation of linear harmonic oscillator is

$$m \frac{d^2x}{dt^2} + kx = 0, \quad \frac{dx}{dt} = y \quad \dots(8a)$$

These equations may be expressed as

$$\frac{dy}{dt} + \frac{k}{m}x = 0 \quad \text{and} \quad \frac{dx}{dt} = y \quad \dots(8b)$$

So that equation (4) takes the form

$$\frac{dy}{dx} = -\frac{kx}{my}$$

$$\text{or} \quad mydy + kxdx = 0$$

Integrating

$$\frac{my^2}{2} + \frac{kx^2}{2} = C \quad \dots(9)$$

The arbitrary constant C is determined by requiring that $x = x_0$ at $y = 0$; this fixes the total energy of E of the oscillator and C is given by

$$C = 0 + \frac{kx_0^2}{2} = E \quad \dots(10)$$

Now equation (9) may be expressed as

$$my^2 + kx^2 = 2E$$

$$\Rightarrow \frac{x^2}{2E/k} + \frac{y^2}{2E/m} = 1 \quad \dots(11)$$

This equation has the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \dots(12)$$

This is well known equation of an ellipse with centre at origin, semi-major axis $a = \sqrt{\frac{2E}{k}}$ and semiminor axis $b = \sqrt{\frac{2E}{m}}$.

The phase trajectory of differential equation (8) is a definite ellipse for a given value of total energy E . For different values of total energy E of system, equation (9) fixed entire family of ellipses. In other words the different ellipses in phase plane correspond to different values of total energy of the system. The equation $y = \dot{x}$ indicates that the representative point $P(x, y)$ traverses the phase trajectory in clockwise sense. The centre of any ellipse $x = 0, y = 0$ is a singular point of the system. This singular point is enclosed by all Trajectories. A singular of this type which is enclosed by all trajectories and approached by none is called a 'vortex point'. The vortex point is a position of stable equilibrium of the system.

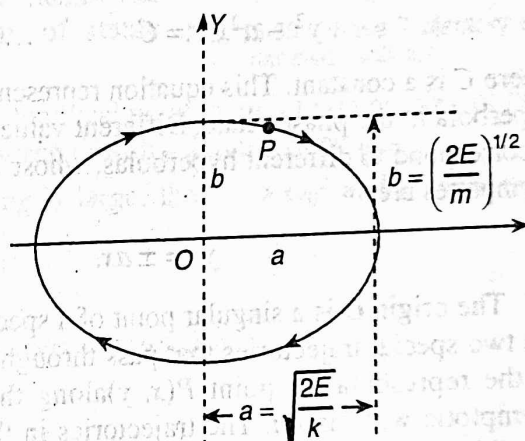


Fig. 15.53. Phase trajectory of linear oscillator

If $x = x_0, y = y_0$ at $t = 0$, the time integral of equations (8) gives

$$\left. \begin{aligned} x &= \frac{y_0}{\omega} \sin \omega t + x_0 \cos \omega t \\ y &= y_0 \cos \omega t - \omega x_0 \sin \omega t \end{aligned} \right\} \text{with } \omega = \sqrt{\frac{k}{m}} \quad \dots(13)$$

These are the parametric equations of ellipse (11).

The period of revolution of representative point

$$T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{m}{k}} \quad \dots(14)$$

2. **A periodic Motion.** Consider a particle of mass m being repulsed from the origin along X-axis by a force proportional to its distance from the origin, i.e., $F \propto x$ or $F = kx$, where k is a constant.

From Newton's second law $F = m \frac{d^2x}{dt^2}$, therefore equation of motion becomes

$$m \frac{d^2x}{dt^2} = kx$$

or $\frac{d^2x}{dt^2} = \frac{k}{m}x$ or $\frac{d^2x}{dt^2} = a^2x$... (15)

where $a^2 = \frac{k}{m}$

Equation (15) is equivalent to two differential equations of first order as

$$y = \frac{dx}{dt}, \quad \frac{dy}{dt} = a^2x$$

Dividing these equations the phase trajectories are given by the equations

$$\frac{dy}{dx} = \frac{a^2x}{y}$$

$$\Rightarrow ydy - a^2xdx = 0$$

Integrating we get

$$y^2 - a^2x^2 = C \quad \dots (16)$$

where C is a constant. This equation represents a hyperbola in the phase plane. Different values of C correspond to different hyperbolas, whose two asymptotes are

$$y = \pm ax.$$

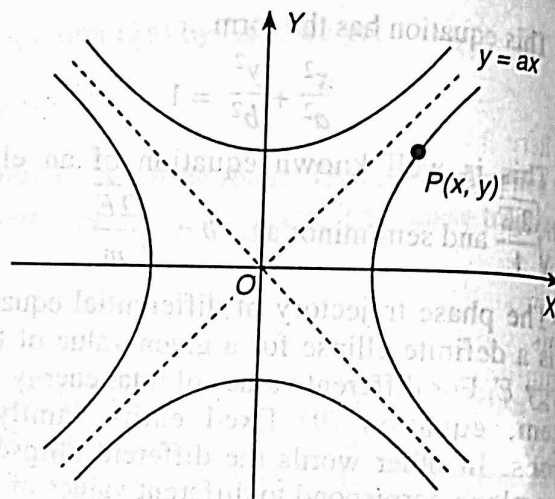


Fig. 15-54

The origin O is a singular point of a special type called the *saddle point*. In this case, there are two special trajectories that pass through the singular point (for which $C = 0$). The motion of the representative point $P(x, y)$ along the trajectories that approach the saddle point is asymptotic with time t . The trajectories in the neighbourhood of a saddle point represent the possible motions that occur in the neighbourhood of a point of unstable equilibrium.

3. Motion of a Damped Oscillator

The differential equation of a damped harmonic oscillator with coefficient of damping ' b ' is given by

$$m\frac{d^2x}{dt^2} = -b\frac{dx}{dt} - kx$$

where k is spring factor. This equation may be written as

$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = 0$$

or

$$\frac{d^2x}{dt^2} + 2r\frac{dx}{dt} + \omega_0^2x = 0 \quad \dots (17)$$

where $2r = \frac{b}{m}$ and $\omega_0^2 = \frac{k}{m}$

This equation is equivalent to two differential equations of first order given by

$$y = \frac{dx}{dt}$$

$$\frac{dy}{dt} = -(2ry + \omega_0^2x)$$

and

Dividing these equations, we get the equation of phase trajectory as

$$\frac{dy}{dx} = -\frac{2ry + \omega_0^2 x}{y} \quad \dots(19)$$

Case (i) Underdamped case

If $r^2 < \omega_0^2$, the motion of particle is damped oscillatory motion. Substituting $\omega = \sqrt{\omega_0^2 - r^2}$

The general solution of equation (17) is

$$\left. \begin{aligned} x &= Ae^{-rt} \cos(\omega t + \phi) \\ y &= \dot{x} = -Ae^{-rt} [r \cos(\omega t + \phi) + \omega \sin(\omega t + \phi)] \end{aligned} \right\} \dots(20)$$

where A and ϕ are arbitrary constants.

These equations represent family of spirals; one spiral is shown in Fig 15-55. The origin O is a singular point called the *focal point*. The representative point $P(x, y)$ spirals to approach the focal point at origin in an asymptotic manner. In this case the focal point at origin is a point of stable equilibrium.

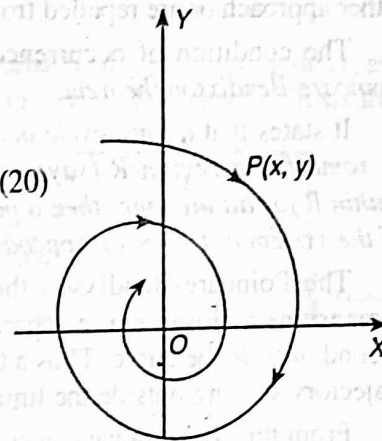


Fig. 15-55. Phase Trajectory of a damped oscillator

The representative point spirals to approach the focal point in an infinite number of times about it, so we may conclude that it does not approach O with a definite direction.

Case (ii) Overdamped motion. If damping is large, then $r^2 > \omega_0^2$ and motion is not oscillatory and is called overdamped motion.

Let $\omega = \sqrt{r^2 - \omega_0^2}$

In this case the general solution of equation (17) is expressed as

$$\left. \begin{aligned} x(t) &= Ae^{-rt} \cosh(\omega t + \phi) \\ y(t) &= \dot{x}(t) = Ae^{-rt} [\omega \sinh(\omega t + \phi) - r \cosh(\omega t + \phi)] \end{aligned} \right\} \dots(21)$$

A and ϕ are constants. Equations (21) represents the phase trajectories of overdamped oscillator. A typical trajectory is shown in Fig 15-56 the origin O is a singular point. The phase trajectories are such that the representative point $P(x, y)$ approaches towards the singular point with a definite direction; such a singular point is called a *node* or *nodal point*. The nodal point O is the position of stable equilibrium. The four types of singular points are listed in the following table :

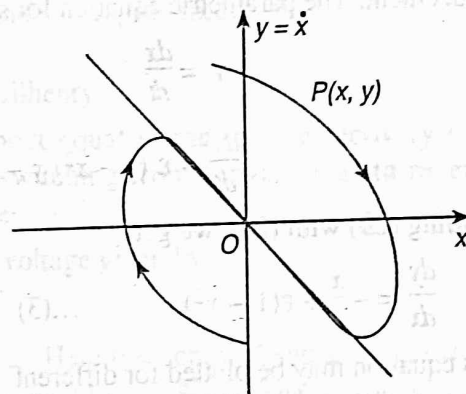


Fig. 15-56. Phase trajectory of overdamped motion

Name of singular point	Type of motion	Type of equilibrium	Approach
1. Vortex point	Oscillatory	Stable	none
2. Saddle point	Aperiodic	Unstable	only along asymptotes
3. Focal point	Damped oscillatory	Stable	with no definite direction
4. Nodal point	Aperiodic	Stable	with a definite direction

15.20. (e) Limit Cycles

In state spaces with two or more dimensions, it is possible to have a periodic behaviour. This behaviour is represented by closed loop trajectories in phase plane. A representative point on one of these loops continues to cycle around that loop for all the time. These loops are called *limit cycles* if the cycle is isolated. The term isolated means that the trajectories nearby either approach or are repelled from the limit cycle.

The condition of occurrence of limit cycle for two-dimension phase state is given by *Poincare-Bendixson theorem*.

It states that a *limit cycle occurs if a long term motion of a representative point is limited to some finite region R (say) such that any trajectory starting within this region (R) stays within R for all the time; then a particular trajectory starting in R either approach a fixed point of the system at $t \rightarrow \infty$ or approaches a limit cycle at $t \rightarrow \infty$.*

The Poincare-Bendixson theorem works only in two dimensions because only in two dimensions a closed curve separates the space into two regions one *inside* the curve and the second *outside* the curve. Thus a trajectory starting inside the limit cycle can never get out and a trajectory starting outside the limit cycle can never get in.

From this theorem we can conclude "*chaotic trajectories in a bounded system cannot occur in phase space of two dimensions.*"

Stable and Unstable Limit Cycles

If we push the system slightly away from the limit cycle and it returns to limit cycle (at least asymptotically), the limit cycle is *stable*; but if the system is repelled from the *limit cycle*, the limit cycle is *unstable*.

Example of limit Cycles Vander Pol Oscillator

The differential equation of Vander Pol oscillator is

$$\frac{d^2x}{dt^2} - \epsilon (1 - x^2) \frac{dx}{dt} + x = 0 \quad \dots(1)$$

where ϵ is a small positive parameter. This system is non-linear and non-conservative and has inbuilt property that it is *self-limiting*. It means that on displacing the system; it itself opposes the displacement. The parametric equation for system are

$$y = \frac{dx}{dt} \quad \dots(a)$$

and

$$\frac{dy}{dt} = \epsilon (1 - x^2) y - x \quad \dots(b)$$

Dividing (2b) with (2a), we get

$$\frac{dy}{dx} = -\frac{x}{y} + \epsilon(1 - x^2) \quad \dots(3)$$

This equation may be plotted for different values of parameter ϵ . When $\epsilon = 0$, the trajectory traversed by a representative point is a *circle* since then equation (3) will become

$$y dy + x dx = 0$$

Integration of which will yield

$$\frac{x^2}{2} + \frac{y^2}{2} = C \Rightarrow x^2 + y^2 = 2C$$

(equation of circle)

When $\epsilon = 0.1$, the trajectory is only slightly different from circle; but when ϵ is

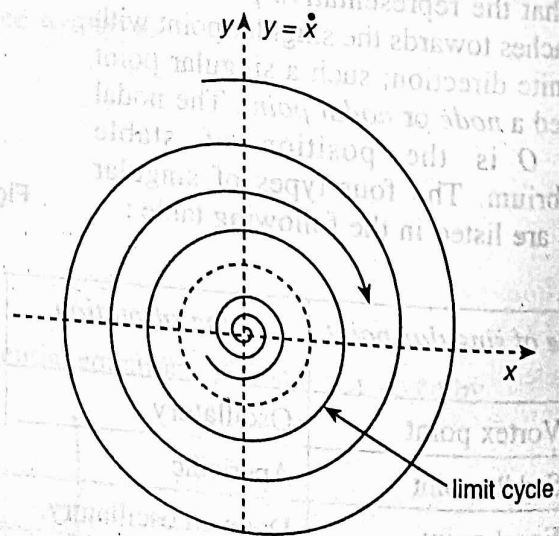


Fig. 15-57. Phase trajectory for $\epsilon = 0.1$

increased, the trajectories differ radically from circle. The Vander Pol oscillator equations tells that for some value of amplitude, the motion neither grows nor decays with time; under this condition the trajectory followed by representative point is a *limit cycle*.

15-20. (f) Route to Chaos

(i) A non-linear Electrical Circuit

A non-linear electrical circuit is shown in Fig. 15-58. It consists of a junction diode D , an inductor L , a dc voltage source V_{dc} and a signal generator S in series. The diode allows the current when forward biased (i.e. when P region is at higher potential than N -region) and does not allow current when reverse biased (i.e. when P region is at lower potential relative to N -region). The valve in human heart works in the same way to regulate the blood flow among the heart-chambers. When the polarity of diode changes from forward to reverse bias there will be a small surge of current before the diode switch fully closes. This time is called *reverse recovery time*. For junction diodes it is a few microseconds.

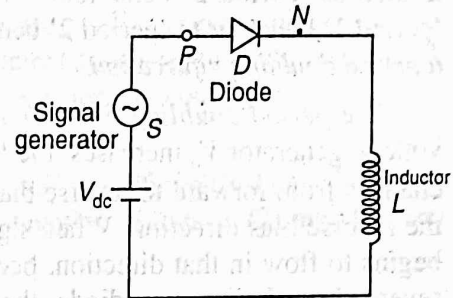


Fig. 15-58

The *capacitance* of junction diode is about 100 pico-period. Due to this capacitance the junction diode stores electrostatic energy temporarily. The inductance in the circuit plays the role of inertia, i.e., it opposes the rate of change of current in electrical circuit without the inductor the behaviour

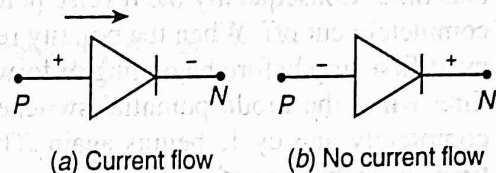


Fig. 15-59.

of current and potential difference in the current are so tightly linked together that the occurrence of chaos is impossible. The presence of inductor has two roles :

- (i) it provides an extra degree of freedom for the chaos to become possible.
- (ii) its combination with diodes capacitance, provides a special frequency for oscillator

usually $\frac{1}{2\pi\sqrt{LC}}$ when L = inductance and C = capacitance.

The inductance used here is typically of 50 millhenry.

When the period of these oscillations is about equal to the reverse recovery time, the current and voltage change so rapidly that the switching from forward bias to reverse bias becomes non-linear and the chaos become possible.

The signal generator provides a time-varying voltage given by

$$v(t) = v_0 \sin 2\pi ft \quad \dots(i)$$

where v_0 is peak voltage, f is frequency of the signal. Here frequency of signal ranges from 20 – 70 kHz; the period of these oscillations being between 14 μ s to 50 μ s. The dc voltage provides a suitable bias voltage to circuit.

Behaviour of Circuit : Period Doubling

To describe the behaviour of circuit we take the amplitude of driving voltage as a variable parameter and call it as a *control parameter*. By changing the amplitude V_0 of the signal generator, we observe the time dependence of potential difference $V_d(t)$ across the diode and current $i(t)$ flowing in diode.

When V_0 is about 0.5 volt (maximum potential difference across diode during its conduction in forward bias), the behaviour of diode is normal, i.e., V_d shows usual half wave rectified waveform and the diode signal has a periodicity equal to the periodicity of the signal generator.

When V_0 is increased to a value between 1 and 2 volts, we observe surprising behaviour. At a well defined voltage V_1 say, the regular array of pulses changes into an alternating series. Fig. 15-60 (a) shows regular pulses while Fig. 15-60 (b) shows that sequence of peaks in V_d is still periodic with a period twice the driving voltage. This behaviour is termed as 'period 2' behaviour. The change from 'period 1' behaviour to 'period 2' behaviour is called a *period doubling bifurcation*.

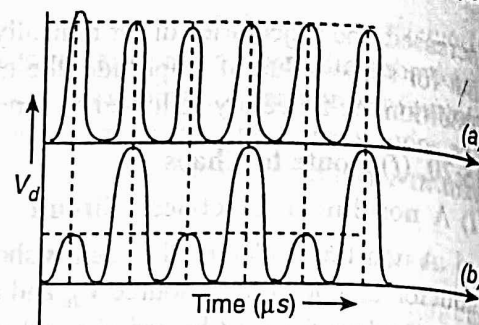


Fig. 15-60.

The *period doubling behaviour* may be explained as follows : When amplitude of signal voltage generator V_0 increases, the forward current in circuit increases. When polarity of diode changes from forward to reverse bias, the diode has time to stop the current flow completely in the reverse bias direction. When signal changes polarity to forward bias the diode, the current begins to flow in that direction, becomes maximum. Now when the polarity again changes to reverse bias the junction diode, the current does not stop completely in the time between the polarity reversals and there is flow of current in reverse bias direction during most of the reverse bias time. Consequently the reverse potential drop is lower than the value when the current had completely cut off. When the polarity returns to forward bias, the reverse-bias direction current must first stop before beginning of forward current. Hence the forward current is smaller. This time when the diode potential switches to reverse bias, the diode switch has time to close completely and cycle begins again. Thus for small amplitudes the diode switch closes each time, so each surge of forward bias current is the same.

When V_0 increases above V_1 , one of the peaks of in the sequence becomes larger and the other smaller. This means another period doubling bifurcation occurs. In this situation the output consists of four distinct peak sizes with a repetition period four times of the generator signal. If we continue to increase V_0 , further period doubling bifurcation continues, i.e., it leads to period-8, period-16 and so on. When voltage becomes sufficient (say V_∞), the sequence of peaks becomes completely erratic, this is the situation, called CHAOS.

The occurrence of chaos may be experimentally verified by observing the divergence of nearby trajectories on the oscilloscope screen. The behaviour of diode circuit may be summarised as follows :

- (i) When parameters of system are slowly changed, there is a sudden change in qualitative behaviour of system. For example, when voltage V_0 is changed slowly the 'period-1' changes suddenly to 'period-2'.
- (ii) When parameter is changed continuously, the regular and periodic behaviour of system changes to a periodic and chaotic behaviour. The dependence of behaviour of system becomes nearly independent of the periodic behaviour of the impressed forces (e.g. signal's period).
- (iii) Chaos can be observed by observing the divergence of nearby trajectories.

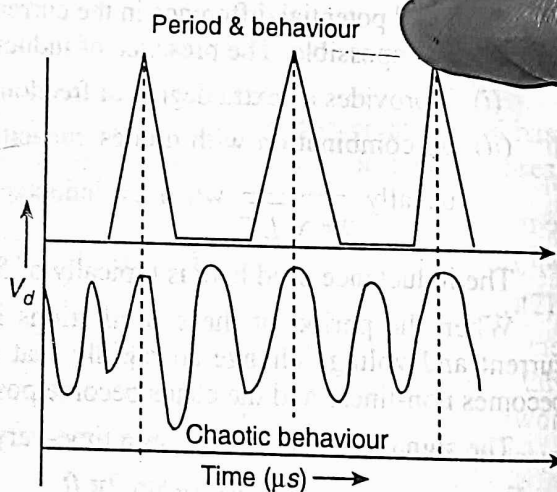


Fig. 15-61. In chaotic behaviour peaks are of unequal size and are non-periodic.

2. Logistic Map Model

Logistic map model is a simple model to describe the chaotic behaviour of the growth of biological population. Consider a species such as mayflies whose individuals are born and die in the same season. If N_0 is the initial number of species, then the number of species (μ_1) after 1 year may follow simple relation

$$N_1 = A N_0 \quad \dots(1)$$

where A is constant which depends on environment, i.e., food supply, water, weather conditions, etc. If $A > 1$, then the number of species will increase. If $A < 1$, the number will decrease. If A is same for next generations, then the population will continue to increase. As the population goes on increasing then there will be insufficient food and so it will be limited. This fact may be taken into account by considering an additional term to equation (1)

$$\text{i.e.} \quad \mu_1 = A\mu_0 - BN_0^2 \quad \dots(2)$$

where $B \ll A$, so the second term will be important only if N is sufficiently large. Negative sign indicates that the second term tends to decrease the population. Equation (2) may be used repeatedly to find the value of N in subsequent years.

$$\text{i.e.} \quad \left. \begin{aligned} N_2 &= AN_1 - BN_1^2 \\ N_3 &= AN_2 - BN_2^2 \end{aligned} \right\} \quad \dots(3)$$

According to equation (2), there is a maximum possible population number.

In order to have $N_{n+1} > 0$, $N_n < \frac{A}{B}$; so $N_{\max} = \frac{A}{B}$.

Introducing a variable

$$x_n = \frac{N_n}{N_{\max}}, \text{ equations (2) may be expressed as}$$

$$x_{n+1} = A x_n (1 - x_n) = f_A(x) \quad \dots(4)$$

where x_n is the population as a fraction of $N_{\max}$ in n th year, and the function $f_A(x)$ is called the *iteration function* because population fraction x in subsequent years may be found by iterating the mathematical operation in (4). This function is called the *iterated map function* or *logistic map function*. The diagram representing $f_A(x)$ against parameter A is called the *logistic map*.

The sequence of values of x generated by iteration process is called the *trajectory* or *orbit*. The trajectories for $A = 1, 2, 3, 4$ are shown. Obviously first few points of trajectory depend on starting value of x .

Fixed Points. Some starting points are different from the others. For example if we take $x_0 = 0$, then $f_A(x_0) = 0$ and the trajectory remains at $x = 0$ for all subsequent iterations

For value of x (say x^*), which gives

$$x_A^* = f_A(x_A^*) \quad \dots(5)$$

is called the *fixed point* of logistic map.

For logistic map function [given by equation (4)], there are two fixed points

$$x_A^* = 0 \quad \text{and} \quad x_A^* = 1 - \frac{1}{A} \quad \dots(6)$$

For $A < 1$, $x_A^* = 0$ is the only fixed point in the range of our biological model. For $A > 1$, both fixed points lie in range $x = 0$ and 1.

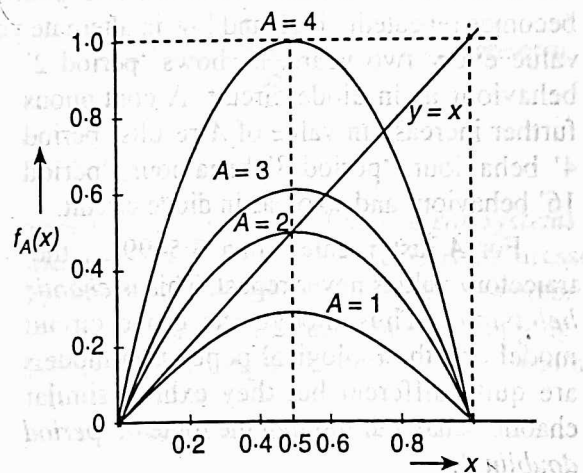


Fig. 15-62. $f_A(x)$ plotted as a function of x for different values of A . The diagonal line is $y = x$.

Importance of Fixed Points. Fig. ... represents $f_A(x)$ versus x plot for values of $A = 1, 2, 3, 4$. The straight line represents $y = x$. The positions where the straight line crosses the curve for $f_A(x)$ represent fixed points of map function since then $f_A(x) = x$. For $A < 1$, there is only one fixed point and for $A > 1$, there are two fixed points in the range $x = 0$ and 1.

It is an important consequence that a trajectory starting from some value of x (different from zero) say x_0 approaches x_0 for $A < 1$. The method is as follows. We choose a point x_0 on X -axis; draw a vertical line x_0a intersecting the $f_A(x)$ curve. The intersection point 'a' determines the value of x_1 . Now we draw a line ab parallel to x -axis. This line intersects the straight line $y = x$ at b . From b we draw vertical line intersecting the $f_A(x)$ curve at c . This intersection point with $f_A(x)$ curve determines x_2 . Continuing this process we arrive at final value $x = 0$.

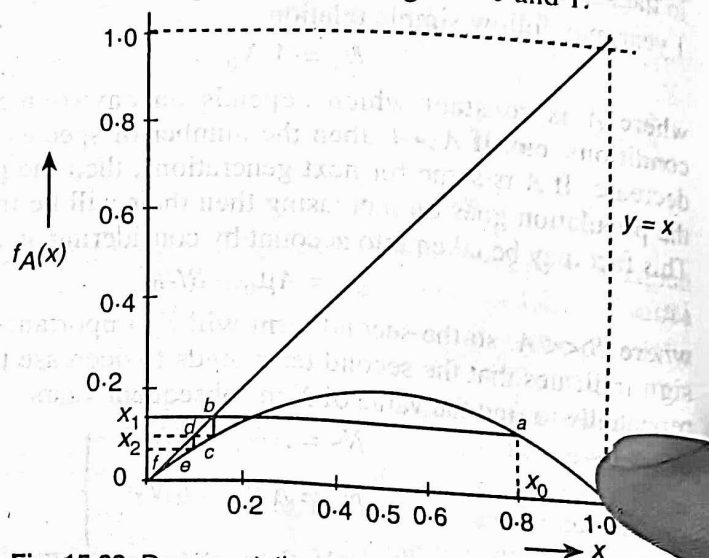


Fig. 15.63. Representation of iteration of $f_A(x)$ starting from $x_0 = 0.8$ $A < 1$

Thus all trajectories (for starting values of x between 0 and 1 for $A < 1$) approach the final value $x = 0$; therefore the fixed point $x = 0$ is called the *attractor* for these trajectories. The interval $0 < x \leq 1$ is called the *basin of attraction* for the attractor; because any trajectory starting in this range approaches fixed point $x = 0$ as the number of iterations increases. The consequence of biological model is that the population ($x \rightarrow 0$) goes on decaying with increase of the number (n) of seasons and finally becomes zero when $n \rightarrow \infty$.

When the parameter $A > 1$, the attracting fixed point is $x_A^* = 1 - \frac{1}{A}$; now $x^* = 0$ has turned to a repelling fixed point.

In this simple model, the surprise occurs when $A > 3$. In this case the trajectory values oscillate back and fourth between two points. In biological terms, the population fraction becomes repeatedly high and low in alternate years. As the population fraction attains the same value every two years, it shows 'period 2' behaviour as in diode circuit. A continuous further increase in value of A results 'period 4' behaviour, 'period 8' behaviour, 'period 16' behaviour and so on as in diode circuit.

For A just greater than 3.5699..., the trajectory values never repeat. This is *chaotic behaviour*. Thus though the diode circuit model and the biological population models are quite different but they exhibit similar chaotic behaviour through the route of 'period doubling'.

Bifurcation diagram for logistic map function is shown in Fig. 15.64.

For $A < 1$, attractor is $x = 0$

For $1 < A < 3$, the attractor is $x = 1 - \frac{1}{A}$

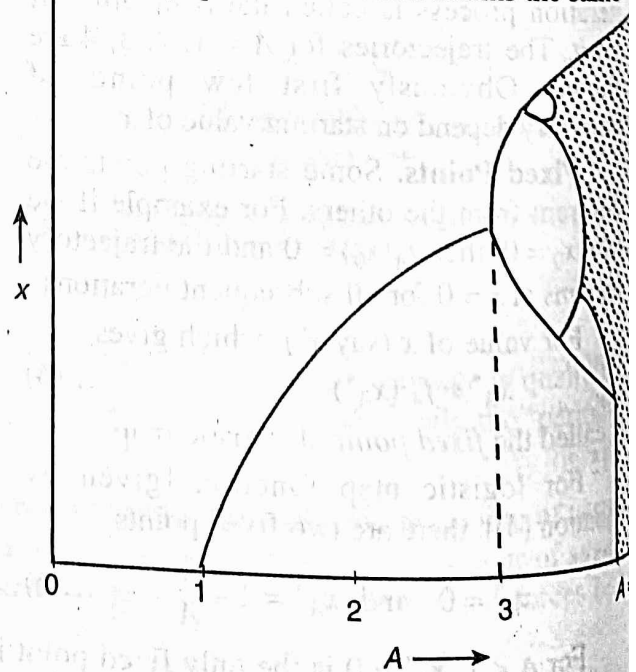


Fig. 15.64

At $A = 3$, bifurcation (period 2) begins. The shaded region indicates the occurrence of chaos.

15.20. (g) Strange Attractors

As dissipative system evolves in time, the trajectories in phase occupy a final state; which may be a point or a curve or an area. This final point curve is called an *attractor* of this system. For dissipative systems the properties of these attractors determine the dynamical properties of the system for its long-term behaviour. For chaotic behaviour of system, three or more finite space dimensions are needed.

For dissipative systems, the trajectories eventually approach an attractor. We are concerned with the behaviour of trajectories within the attracting region of state space. The chaotic behaviour of system is characterised by the divergence of nearby trajectories in state space. As a function of time, the separation between two nearby trajectories increases exponentially, at least for very short duration. This restriction is essential because we are considering the systems whose trajectories stay within some bounded region of state space.

For a system not to blow up and show chaotic behaviour, the following three conditions must be satisfied.

- (i) Different trajectories should not intersect.
- (ii) Trajectories should be in bounded region.
- (iii) Nearby trajectories diverge exponentially.

These conditions cannot be satisfied simultaneously in one or two dimensions. Such trajectories can be drawn in three dimensions. These trajectories can remain within some bounded region by intertwining and wrapping around each other without repeating themselves exactly (Fig. 15.65).

Obviously the geometry associated with such trajectories is strange; therefore such attractors are called *strange attractors*. In terms of fractal dimensions, the definition of strange attractors is as follows :

If an attractor for a dissipative system has a non-integer dimensions, then the attractor of the system is said to be strange attractor.

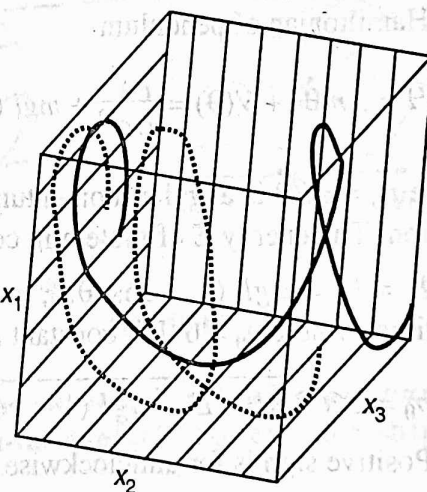


Fig.. 15.65. Trajectories in three dimensional state space

15.20. (h) Butterfly Effect

The effect of divergence of nearby trajectories on the behaviour of non-linear systems is called the *butterfly experiment*. The effect was stated by E.M. Lorentz in 1972. Lorentz stressed on the point that if atmosphere displays chaotic behaviour with divergence of nearby trajectories or sensitive dependence on initial conditions, then even a small effect, such as flapping of butterfly's wings may affect the weather conditions; so our long-term prediction of weather becomes meaningless.

15.20. (i) Invariant Tori

The Hamilton's equation in terms of action variables (J_k) and angle variables (θ_k) can be expressed as

$$\dot{\theta}_k = \frac{\partial H(J, \theta)}{\partial J_k} \text{ and } \dot{J}_k = -\frac{\partial H(J, \theta)}{\partial \theta_k} \quad \dots(1)$$

with $k = 1, 2, \dots, N$

If these equations are satisfied, then the variables (θ, J) are related to variables (q, p) by a canonical transformation.

If action variables are constants of motion, *i.e.*

$$\dot{J}_k = 0, \text{ for } k = 1, 2, \dots, N, \quad \dots(2)$$

then the Hamiltonian of the system is said to be integrable. The term integrable comes from the notion that the action J_k can be expressed as an integral over the motion of the system and the corresponding equation for θ_k can easily be integrated.

For a system having an *integrable Hamiltonian*,

$$\dot{\theta}_k = \frac{\partial H}{\partial J_k} = \omega_k(J) \quad \dots(3)$$

where $\omega_k(J)$ is angular frequency of motion and depends on all values of J_k 's. As J_k 's are independent of time, ω_k 's are also independent of time. Therefore integration of equation (3) gives

$$\theta_k = \omega_k t + \theta_k(0) \quad \dots(4)$$

where $\theta_k(0)$ is constant of integration.

To arrive at an important result we consider the one-dimensional motion of a pendulum.

Hamiltonian of pendulum

$$H = \frac{1}{2} m \dot{\theta}^2 + V(\theta) = \frac{p_\theta^2}{2ml^2} + mgl(1 - \cos \theta) \quad \dots(5)$$

where $p_\theta = ml^2 \dot{\theta}$ is angular momentum associated with the rotation. The energy E of system is constant, so

$E = H = mgl(1 - \cos \theta_0)$, where θ_0 is angular amplitude where $p_\theta = 0$. For constant E , equation (5) gives

$$p_\theta = \pm \sqrt{2ml^2 \{E - mgl(1 - \cos \theta)\}} \quad \dots(6)$$

Positive sign is for anticlockwise and negative for clockwise motion.

The action associated with a closed trajectory is related to phase space area associated with the trajectory, we can write

$$J = \oint p_\theta d\theta \Rightarrow \oint \left[\sqrt{2ml^2 \{E - mgl(1 - \cos \theta)\}} \right] d\theta \quad \dots(7)$$

This is elliptical integral. For relatively small values of energy, the phase space trajectories are ellipses centered at the origin.

If we compare equations (3) and (4) with the results of one-dimensional pendulum, we see that an integrable system with N -degrees of freedom is equivalent, in terms of action-angle variables to a set of N uncoupled oscillators. As there are N constants of motion for an integrable system with N -degrees of freedom, the trajectories in the state space are highly constrained. For example an integrable system with two degrees of freedom has trajectories residing in four-dimensional phase space. This surface is the surface of a torus tube of a bicycle tyre.] The trajectories are characterised by two frequencies

$$\omega_1 = \frac{\partial H}{\partial J_1}, \quad \omega_2 = \frac{\partial H}{\partial J_2} \quad \dots(8)$$

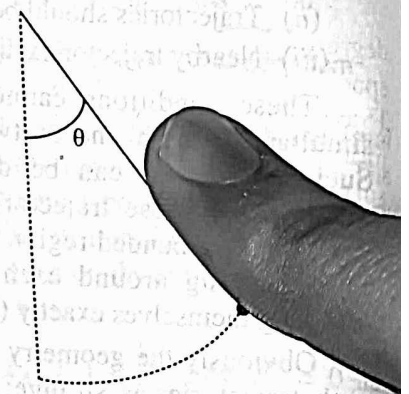


Fig. 15-66

Generalising the result we can say that a trajectory for an integrable system with N degrees of freedom is constrained to N -dimensional surface of a torus which resides in original $2N$ -dimensional phase space. As the motion is confined to these surface for all time, these tori are often called *invariant tori*.

15-20. (j) KAM Theorem

As the behaviour of an integrable Hamiltonian system is always periodic or quasi-periodic; therefore an integrable system *cannot* display chaotic behaviour. The chaotic behaviour is exhibited by systems only if the Hamiltonian of system is slightly non-integrable. Now we have to think how the behaviour of the system deviates from that of an integrable system with increase of value of non-integrability of system. For this we have to visualize the trajectories for nonintegrable systems, because a nonintegrable system has at least two degrees of freedom. For an integrable system, the trajectories move on a two dimensional surface of a torus and be either periodic or quasi-periodic. For system having two degrees of freedom, the phase space is four-dimensional. If the system is non-integrable, then the trajectories can move on a three dimensional surface of this four-dimensional phase space because the energy is conserved and hence still constrains the trajectories. This three-dimensional motion permits the possibility of chaos.

For an integrable system having two degrees of freedom, one can imagine the phase space containing a set of nested tori (Fig. 15-67). For fixed values of two constants of motion, the trajectories are confined to the surface of one of the tori. When the system become non-integrable, then the trajectories begin to move off the tori. That is we can say that the tori are destroyed. To tell about the specific survival of tori Kolmogorov-Arnold-Moser stated a theorem called KAM theorem after initials of three scientists. It states *"some phase space tori, in particular, those associated with quasi-periodic motion with an irrational winding number survive if an integrable system is made slightly non-integrable"* [however the shape of tori may get slightly disorted].

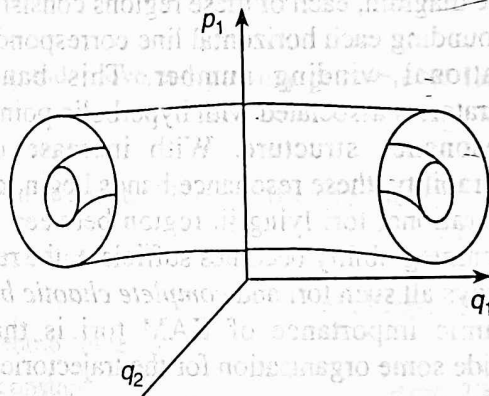


Fig. 15-67

This theorem may be stated in a different ways as follows :

Suppose the Hamiltonian of the originally integrable system can be written as the function of action variables alone, i.e., $H = H_0(J)$; the system is made non-integrable, by adding a smaller term dependent on action and angle variables, i.e., for non-integrable system

$$H = H_0(J) + \epsilon H_1(J, \theta) \quad \dots(1)$$

Where ϵ is a small parameter which controls the size of non-integrability term. The second term in equation is called the perturbation term and is evaluated by perturbation theory of Quantum Mechanics.

The KAM theorem states that for $\epsilon \ll 1$ (so that the system is almost integrable), the tori with irrational ratios of frequencies associated with the action will survive. These surviving tori are called KAM tori. As ϵ increases, the tori dissolve one by one and the last surviving torus has winding number equal to the 'most irrational' of the irrational numbers called the *Golden Mean*.

In terms of winding number ω , the KAM theorem states that the tori will survive for a given amount of non-integrability if the winding number ω satisfies the following inequality

$$\left| \omega - \frac{m}{n} \right| > \frac{g(\epsilon)}{n^{2.5}}$$

where the ratio $\left(\frac{m}{n}\right)$ of positive numbers m and n has its simplest form. The factor $g(\epsilon)$ which increases with the amount of non-integrability, is same for all values of $\frac{m}{n}$. We note that the tori with winding number ω close to rational fractions with small denominators (e.g. $\frac{1}{4}$ or $\frac{1}{3}$) will be the first to dissolve, while the tori with winding numbers such as Golden Mean which are faster from low denominator rational fractions, will survive to larger values of ϵ .

The dynamical importance of KAM tori is that they continue to provide some organisation for the trajectories in phase space.

The tori-dissolving may be visualised as follows :

Suppose we imagine a regions of size $\frac{g(\epsilon)}{n^{2.5}}$ around each rational fraction $\frac{m}{n}$. Any torus with an irrational winding that lies within one of these regions will dissolve. If we consider the corresponding action-angle diagram, each of these regions consists of a horizontal band surrounding each horizontal line corresponding to a J -value with a rational winding number. This band bounded by the separatrices associated with hyperbolic points of J -value is called a resonance structure. With increase of amount of non-integrability, these resonance bands begin to overlap and destroy the irrational tori lying in region between them. When amount of nonintegrability becomes sufficient, the resonance-overlapping destroys all such tori and *complete chaotic behaviour* occurs. The dynamic importance of KAM tori is that they continue to provide some organisation for the trajectories in the phase space.

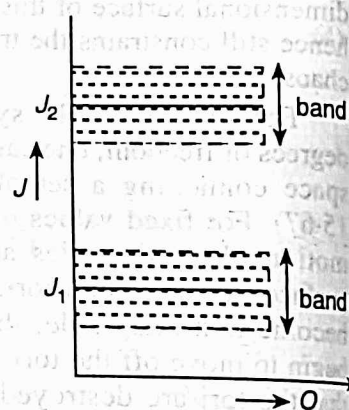


Fig. 15-68

EXERCISES

Short Answer Questions

1. State the theorem of conservation of linear momentum.
2. State the theorem of conservation of angular momentum.
3. State the theorem of conservation of energy.
4. State the work-energy theorem.
5. What are generalised coordinates ? Give examples. (Kanpur 2007)
6. Define generalised coordinates and conjugate momentum. (Kanpur 2010, 07)
7. What do you mean by constraints ?
8. Distinguish between holonomic and non-holonomic constraints.
9. What are transformation equations relating cartesian coordinates and generalised coordinates ?
10. What is configuration space ?
11. What is the principle of Virtual Work ? (Kanpur 2007, 06)
12. State D'Alembert's principle. (Kanpur 2006)
13. What is conservative system ? Write Lagrange's equation for conservative system.
14. Write Lagrange's equation for a simple pendulum.
15. What is generalised momentum ? Express it in generalised coordinates.

16. Show that generalised momentum $p_k = \frac{\partial L}{\partial \dot{q}_k}$.
17. What is a cyclic coordinate ? Show that if a coordinate is cyclic, the corresponding generalised momentum is constant. (Agra 2009)
18. Show that the kinetic energy of a system of particles may be expressed as

$$T = T_0 + \sum_k a_k \dot{q}_k + \sum_{k,l} a_{kl} \dot{q}_k \dot{q}_l$$
 where T_0 , a_k , a_{kl} are independent of generalised velocities.
19. Define phase space.
20. What is physical significance of Hamiltonian ?
21. Derive Hamilton's equations. (Agra 2010)
22. Write Hamiltonian of a simple pendulum.
23. Write Euler-Lagrange's differential equation.
24. Show that the shortest distance between two points in a plane is a straight line.
25. State Hamiltonian principle.
26. State and explain principle of least action. (Kanpur 2007, Agra 2007)
27. Express principle of least action in Jacobi form.
28. State Fermat's principle of geometrical optics and prove it by using principle of least action.
29. Define a central force.
30. Demonstrate how a two body central force motion is converted to one body problem. (Kerala 2003)
31. What are the first integrals of motion under a central force ?
32. State Kepler's laws of planetary motion. (Kanpur 2007)
33. Name the orbit of planet under inverse square central force law.
34. Show that areal velocity of a planet remains constant. (Agra 2007)
35. State Virial theorem.
36. What do you mean by scattering cross section ? Discuss Rutherford scattering.
37. What do you mean by laboratory and centre of mass systems ? (Kanpur 2007)
38. Obtain the trilinear invariant condition for the transformation to be canonical ? (Kanpur 2007)
39. What do you mean by canonical transformations ?
40. Find the condition for a transformation to be canonical. (Kanpur 2006)
41. What are canonical transformation ? Solve the problem of a simple harmonic oscillator unit's canonical transformations. (Meerut 2010)
42. Define Lagrange's bracket. Show that it is invariant under canonical transformation. (Kerala 2003)
43. What are Poisson brackets ? (Agra 2009)
44. State and prove Jacobi-identity of Poisson brackets. (Kanpur 2007)
45. Show that Poisson bracket of two arbitrary functions remains invariant under canonical transformations.
46. Write equation of motion in Poisson bracket form.
47. Show that a function is an integral of motion if it does not depend on time explicitly and its Poisson bracket with Hamiltonian vanishes.
48. Show that the Poisson bracket of two integrals of motion is itself an integral of motion. (Agra 2009)
49. Derive Hamilton Jacobi equation. Show that the Hamilton's principal function 'S' is given by $S = \int L dt + \text{constant}$. (Kanpur 2006, Kerala 2002)

50. What is the physical significance of Hamilton's characteristic function ?
51. What are action and angle variables ?
52. Write Euler's equations of motion.
53. What is the condition for the stability of a spinning top ?
54. Obtain Hamilton's principal function for one dimensional harmonic oscillator. (Kerala 2003)
55. A particle moves on a force field $\vec{F} = \phi(r) \vec{r}$. Show that its angular momentum is conserved. (Kerala 2003)
56. What is the purpose served by the canonical transformation ? Can you say that Hamilton-Jacobi theory is a canonical transformation ? (Kerala 2002)
57. What do you mean by non-linear dynamics ?
58. What do you mean by 'chaos' ?
59. State pendulum equation in linear and non-linear forms.
60. What do you mean by 'Autonomous systems' ?
61. What do you mean by limit cycles ?
62. Differentiate between stable and unstable limit cycles.
63. State the equation of Vander Pol Oscillator. Why is it called self-limiting ?
64. What do you mean by 'period doubling' ?
65. What is meant by logistic map ? (Calicut 2002)
66. Explain the term strange attractor. (Calicut 2003)
67. What is meant by butterfly effect in 'chaos'. (Calicut 2003)
68. Explain the use of logistic map in the study of chaos. (Calicut 2003)
69. Write a note on chaos in dissipative systems. (Calicut 2002)
70. What do you mean by fixed points in logistic map function ? What is their importance ?
71. What do you mean by invariant tori ?
72. State KAM Theorem.
73. Using the fundamental Poisson brackets find the values of α and β , for which

$$Q = q^\alpha \cos \beta p$$

$$P = q^\alpha \sin \beta p$$
 represent canonical transformation. (Kanpur 2007)
74. Show that the work done by constraint forces on a rigid body is zero. (Kanpur 2006)
75. If $[\alpha, \beta]$ are Poisson's bracket, then prove that

$$\frac{\partial}{\partial t} [\alpha, \beta] = \left[\frac{\partial \alpha}{\partial t}, \beta \right] + \left[\alpha, \frac{\partial \beta}{\partial t} \right] \quad [\text{Kanpur 2006}]$$

Long Answer Questions

1. (i) What do you mean by constrained motion ? Distinguish clearly between
 - (a) holonomic and non-holonomic constraints. (Rohilkhand 1998)
 - (b) Scleronomic and rheonomic constraints
- (ii) In each of the following cases state whether the constraint is holonomic or non-holonomic, giving a reason for your answer :
 - (a) A rigid body in which the distance between any two particular particles is fixed.
 - (b) The molecules moving inside a gas container.
 - (c) A particle sliding down a sphere from a point near the top under the influence of gravity.
 - (d) A double pendulum oscillating in a vertical plane.

- (e) Two masses connected by a massless inextensible string passing over a smooth massless pulley. (Rohilkhand 1988)

2. (a) What are constraints? What do you mean by holonomic and non-holonomic constraints. Explain with examples.

(Kanpur 2007, Garhwal 1998, Rohilkhand 1986, Meerut 1997)

(b) What are generalised coordinates? What is the advantage of using them?

(Rohilkhand 2000, 1987, Agra 2010)

(c) Obtain expressions for generalised displacement, generalised velocity, generalised acceleration and generalised force. (Kanpur 2006)

(d) State and explain D'Alembert's principle and derive Lagrange's equations from it.

(Kerala 2002, Garhwal 1999, Rohilkhand 2005, 2003, Meerut 1997, Agra 2010, 09, 2004, 1991)

3. Derive Lagrange's equation in the form

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_k} \right) - \frac{\partial T}{\partial q_k} = Q_k$$

where symbols have usual meanings.

(Rohilkhand 1986)

4. What are holonomic constraints? Illustrate your answer with examples. What difficulty arises in the solution of dynamical problems involving constraints and how are they overcome in cases of holonomic constraints. (Rohilkhand 1991, Meerut 1998)

5. (a) Derive Lagrange's equations of motion from D'Alembert's principle for a holonomic conservative system. (Garhwal 1999, Rohilkhand 1993, 89, 87, Agra, 2000, 96, 93,

Kanpur 2007, Meerut 2001, 1993, 87, 82)

(b) A double pendulum vibrates in a vertical plane. Write the Lagrangian of the system and derive the equations of motion. (Rohilkhand 1998, 83, 81)

6. (a) Derive Lagrange's equation of motion for non-conservative system. (Rohilkhand 2000) or Derive Lagrange's equation for a system containing dissipative forces.

(Rohilkhand 2001)

(b) A particle moves in a plane under the influence of a force, acting towards centre of force, whose magnitude is

$$F = \frac{1}{r^2} \left(1 - \frac{\dot{r}^2 - 2\ddot{r}r}{C^2} \right)$$

where r is distance of particle to the centre of force. Obtain the Lagrangian for the motion in a plane. (Rohilkhand 1985)

7. What is meant by constraints? Derive Lagrange's equation for a nonholonomic system.

(Rohilkhand 1998)

8. What are generalised coordinates and generalized velocities? Set up the Lagrangian for a spherical pendulum. (Rohilkhand 1994)

9. Discuss the superiority of Lagrangian approach over Hamiltonian approach. Show that

$$\sum_{i=1}^n p_i \dot{q}_i - L(q, \dot{q}, t) = \text{constant.}$$

(Rohilkhand 1994)

10. Derive Lagrange's equation of motion. A particle of mass m moves in a two dimensional potential

$$V(x, y) = -(k_1 x^2 + k_2 y^2)$$

Write down (i) the Lagrangian, (ii) the Hamiltonian, (iii) the Lagrange's equation of motion.

(Rohilkhand 1992)

11. (a) Write the Lagrangian of a system of two particles of unequal masses connected by an inextensible string passing over a small smooth pulley.

Hence obtain equation of motion.

(Meerut 1992, 87; Agra 1983)

- (b) Apply Lagrange's equation to determine the motion of a bead sliding on a uniformly rotating wire in a force-free space. (Kanpur 1995, Agra 1994)
12. (a) Write down the Lagrangian for a system of two particles of mass m_1 and m_2 connected by an inextensible string passing through a hole in a horizontal table such that mass m_2 hangs vertically and m_1 moves on the smooth top of the table. Hence obtain the equations of motion for the system and solve the same. (Bombay 1997)
- (b) Use Lagrange's equation of motion to show that the period of oscillation of a compound pendulum is $2\pi \sqrt{\left(\frac{k^2 + h^2}{gh}\right)}$ where k is radius of gyration, h is the distance of the centre of gravity of the body from the point of suspension. (Agra 1994)
13. (a) Define generalised coordinates, generalised momentum and cyclic coordinates, (Garhwal 1998, Meerut 1993, 87)
- (b) Show that the generalised momentum is conserved only when the corresponding generalised coordinate is cyclic or ignorable. (Meerut 1993, Rohilkhand 1988)
- (c) Hence prove if cyclic coordinate is rotational, the angular momentum remains conserved. (Rohilkhand 1988, Meerut 1983, 80)
- (d) Show that if the Lagrangian does not depend upon time explicitly, the total energy of the system is conserved. (Kanpur 2007, Rohilkhand 1988, Agra 1996; Meerut 1987, 80)
14. What do you mean by cyclic coordinates? Prove that if a coordinate is not present in the Lagrangian, it will not be present in the Hamiltonian. (Meerut 2001)
15. Define generalised coordinates and cyclic coordinates. By giving an example demonstrate the use of cyclic coordinate integrating the integrals of motion. (Rohilkhand 1985, 83)
16. Given a Lagrangian $L = L(q, \dot{q}, t)$, define the canonical conjugate momentum from it. Derive the Hamilton's canonical equations from a Lagrangian. (Garhwal 1997, Rohilkhand 1984)
17. (a) State and prove Hamilton's equations of motion. (b) Use them to obtain the motion of a projectile launched with speed v_0 at an angle α with the horizontal. (Kanpur 2007, Rohilkhand 2009)
18. Derive Hamilton's equations of motion using Legendre transformation method. What is the physical significance of Hamiltonian? (Rohilkhand 1999, 93; Agra 1983)
19. (a) What is Hamilton's function? Explain its physical significance.
- (b) Prove that the Hamiltonian H of a conservative system is equal to total energy of the system. (Agra 2010)
20. Obtain the Hamilton's canonical equations of motion from Lagrangian equations using Legendre transformations. Define a cyclic coordinate and prove that such a coordinate is ignorable in the Hamiltonian formulation. Discuss the physical significance of Hamiltonian. (Agra 1997)
21. Derive Hamilton's canonical equations of motion and explain the physical significance of Hamiltonian function. What are cyclic coordinates? (Meerut 1986, Agra 1999)
22. Obtain an expression for the Lagrangian of a charged particle in an electromagnetic field in the form

$$L = \frac{1}{2}mv^2 + e\mathbf{v} \cdot \mathbf{A} + e\phi$$

where $\mathbf{A}(\mathbf{r}, t)$ and $\phi(\mathbf{r}, t)$ are electromagnetic vector and scalar potentials of the electromagnetic field and $\mathbf{v}$ is the velocity of the particle.

Hence obtain the Hamiltonian and Hamilton's equations for the same particle. (Agra 1994)

23. Derive Euler-Lagrange equations of motion using the method of calculus of variations and hence obtain Lagrange's equation of motion for a system of particles. (Meerut 1983, 77; Agra 1977)

24. Use the technique of the calculus of variations to obtain Euler-Lagrange's equation and show that the minimum distance between two points in a plane is a straight line. (Meerut 2004)
25. Derive Hamilton's equations from the variation principle and give their physical significance. Obtain these equations in spherical coordinates. (Meerut 1982)
26. (a) Determine the differential equations which function $y(x)$ must satisfy that makes the integral

$$I = \int_{x_1}^{x_2} F(y, y', x) \text{ an extremum}$$

- (b) If the Lagrangian of a system is not an explicit function of time, then $\sum p_i \dot{q}_i - L = \text{constant}$. (Rohilkhand 1993)
27. (a) Find the equation of curve joining two points, along which a particle falling from rest under the influence of gravity from higher point to lower point in the least time.
- (b) Show that the Euler-Lagrange equation of the calculus of variation can be written as

$$\frac{d}{dx} \left(y' \frac{\partial f}{\partial y'} - f \right) + \frac{\partial f}{\partial x} = 0$$

where $f = f(y, y', x)$

(Rohilkhand 1994)

28. (a) State and explain Hamilton's principle.
- (b) Derive Lagrange's equation of motion from Hamilton's principle and show that
- (i) generalised momentum conjugate to cyclic coordinate is conserved
- (i) If the Lagrangian is not an explicit function of time, then Hamiltonian is a constant of motion. (Rohilkhand 1991)
- (c) Discuss conservation of angular momentum and associated symmetry operation. (Rohilkhand 2008)
29. Show that
- (a) If the Lagrangian is translationally invariant, then linear momentum is conserved. (Rohilkhand 1993)
- (b) If the Lagrangian is rotationally invariant, then angular momentum is conserved. (Rohilkhand 1993)
- (c) If the Lagrangian is symmetric with respect to time, then energy is conserved.
30. State Hamilton's principle and derive it by differential method. Using Hamilton's principle show that the sphere is the solid figure of revolution which has maximum volume for a given surface area. (Rohilkhand 1982, 79)
31. (a) Derive Hamilton's equations from the variational principle. (Rohilkhand 1999, Agra 1982, 80)
- (b) Deduce Hamilton's principle. How can this principle be used to find the equation of one dimensional harmonic oscillator? (Agra 1982)
32. State and explain Hamilton's principle and derive Lagrange's equation of motion from it for a conservative system. Discuss how the result will be modified if the forces are non-conservative. (Kerala 2002, Meerut 2004, 1996; Agra 1998)
33. What is meant by configuration space? State Hamilton's variational principle and derive Lagrange's equation of motion from it. (Rohilkhand 1989)
34. (a) State and prove the principle of least action and obtain Jacobi's form of principle of least action. (Rohilkhand 1999, Meerut 1992, 91)
- (b) From this principle deduce differential equation of the path of a particle under a central force. (Meerut 1987, Agra 1982, 80)
- (c) If the velocity of a particle in a central orbit varies as $\frac{1}{r^2}$, apply the principle of least action to find the orbit and hence the law of attraction. (Agra 1984)

35. (a) Explain the method of Lagrange's undermined multipliers in deriving the equations of motion for a conservative non-holonomic system, from Hamilton's principle. (Agra 1997, 81)
- (b) Apply this method to solve the problem of a hoop rolling down the inclined plane without slipping. (Agra 1994)
36. (a) A particle of mass m moves under a central force $f(r)$. Show that the general equation of the orbit can be written as

$$\frac{d^2u}{d\theta^2} + u = \frac{m}{l^2 u^2} f(u)$$

where $u = \frac{1}{r}$ and l is the angular momentum.

- (b) Discuss the orbit if the force law is attractive inverse square law.
- (c) If the orbit described under a central force is given by $r = a(1 + \cos \theta)$ with centre at origin; find the law of force. (Meerut 2004, Bharathidasan 1984)
37. (a) Define Lagrangian function
- (b) Discuss Lagrange's equation of motion with examples.
- (c) What is Hamilton's variational principle? Explain.
- (d) Set up the Lagrangian for a simple pendulum and hence obtain equation describing its motion. (Garhwal 1998)
38. Explain how the problem of two bodies moving under the influence of mutual central force can be reduced to one body problem. Discuss the Kepler problem in detail. (Meerut 1991, Rohilkhand 1992, Agra 1990)
39. State and prove Kepler's laws of planetary motion. (Agra 2009, Meerut 2003)
40. (a) Obtain differential equation for a particle undergoing a central force motion and use it to verify Kepler's laws. (Agra 2008, Rohilkhand 2001, 1993, Meerut 1992)
- (b) State which of the Kepler's laws will still be true if the inverse square law of force is no longer valid? (Meerut 1987)
41. Show that if a particle describes a circular path under the influence of an attractive central force directed towards a point on the circle, then the force varies as fifth power of distance. (Rohilkhand 2000, Meerut 1986, 83)
42. Show that the invariance of a Lagrangian of a system under space and time translation symmetries leads to the conservation of its linear momentum and energy respectively. (Rohilkhand 2000)
43. Explain the Hamilton's principle and show that the Lagrange's equations follow from Hamilton's principle for monogenetic systems. (Rohilkhand 2000)
44. Discuss Legendre transformations and the Hamilton's equations of motion. Comment on the non-symmetric structure of these equations and symmetrise them in possible matrix form. (Rohilkhand 2000)
45. (a) Deduce Hamilton's principle from D' Alembert's principle
- (b) Derive Hamilton's canonical equations of motion in generalised coordinates. (Garhwal 1999)
46. A particle moving under central force describes an orbit. Find out its differential equation. If the equation of the orbit is given by $r = 2a \cos \theta$, show that the force varies as fifth power of distance inversely. (Meerut 2000, 1981)
47. State and prove Virial theorem and show that this takes the form $2T + V = 0$, when forces are derivable from potential. (Garhwal 1999, 97; Meerut 1981)
48. Show that if law of central force is an inverse square law of attraction and $E < 0$, the conic is an ellipse. (Meerut 1992)

49. State and prove Hamilton's principle and use it to prove that the shortest distance between two points in space is a straight line joining them. (Meerut 1992)
50. (a) Derive Hamilton's equations from a variation principle. How can this be modified so as to obtain the principle of least action.
(b) A particle of mass m is projected upwards with speed u at an angle α with the horizontal. Find its Hamiltonian and the canonical equations of motion. (Meerut 1993)
51. State and prove the principle of least action. Prove that out of all possible paths between two points, the system for which the kinetic energy is conserved moves along the path for which the transit time is extremum. (Kanpur 2006)
52. What is differential cross-section? Find an expression for it in terms of impact parameter and the scattering angle. Hence derive Rutherford's formula for scattering cross-section under Coulomb field. (Meerut 1992)
53. Discuss the problem of scattering of charged particles by Coulomb field and obtain Rutherford formula for scattering cross-section. (Rohilkhand 2001, 92; Agra 1983, 81)
54. Derive the equations which connect the centre of mass and laboratory coordinate systems. (Rohilkhand 1993)
55. Define scattering cross-section. How will you transform the expression of differential scattering cross-section from centre of mass coordinates to laboratory coordinates. (Rohilkhand 1991)
56. (a) Define canonical transformations and obtain the transformation equations corresponding to all possible generating functions. (Garhwal 1999, Agra 2010, 1982, 80; Rohilkhand 2001, 1999)
(b) Show that the following transformations are canonical ones.

$$(i) P = \frac{1}{2}(p^2 + q^2)$$

$$Q = \tan^{-1} \frac{q}{p}$$

(Rohilkhand 2001, 1992, 79, Meerut 2004, Agra, 1982)

$$(ii) Q = \frac{1}{p} \text{ and } P = qp^2$$

(Agra 2010)

$$(iii) Q = q \tan p,$$

$$P = \log_e \sin p$$

(Rohilkhand 1991)

$$(iv) Q = p,$$

$$P = -q$$

(Rohilkhand 1985)

$$(v) Q = \log \left(\frac{1}{q} \sin p \right); \quad P = q \cot p.$$

(Rohilkhand 1993, 84, 80, Meerut 2000, 1982, 81)

$$(vi) q = \sqrt{2P} \sin Q, p = \sqrt{2P} \cos Q.$$

(Garhwal, 1999, Rohilkhand 1987, Meerut 1992, 91, 87)

57. (a) What are canonical transformations? Discuss briefly why they are considered useful.
(b) Show that the transformation

$$Q = \log(1 + q^{1/2} \cos p)$$

$$P = 2q^{1/2}(1 + q^{1/2} \cos p) \sin p.$$

is canonical. Find the generating function $F(q, Q)$.

(Meerut 1993, Rohilkhand 1999, 86, 85)

58. Write down the Lagrangian of a free particle in rectangular Cartesian coordinates. Identify all the cyclic coordinates. Show that the constants of motion obtained from the equations for the considered from of the Lagrangian are not exactly the same as those which follow from the concept of a free particle. (CSE Meerut 2007)
59. (a) What are canonical transformations? Discuss the condition for a transformation to be canonical. (Rohilkhand 1988, 85, Meerut 2009, 2004, 1983, 82, Agra 1983)
(b) Using $F_1 = m\omega q^2 \cot Q$ as a generating function, obtain an expression for the displacement of a linear harmonic oscillator. (Meerut 1998, 77, Agra 1983, 80)

60. What are canonical transformations ? What values of α and β do the equations $Q = q^\alpha \cos \beta p$, $P = q^\alpha \sin \beta p$, represent a canonical transformation ? What is the form of generating function for this case ?
(Kanpur 2008, Rohilkhand 1994, 88, 82)
61. (a) What are Poisson and Lagrange brackets ? How are they related ? Show that the Lagrange bracket is invariant under canonical transformations.
(Garhwal 1997, Rohilkhand 1988, Agra 1982)
- (b) Prove that for a transformation to be canonical, the fundamental Poisson brackets must be invariant.
(Rohilkhand 2001)
62. (a) Derive equations of motion in terms of Poisson brackets.
(Rohilkhand 1997)
- (b) Give a proof of Jacobi identity and show that the Poisson bracket of two constants of motion is itself a constant of motion.
(Rohilkhand 1999, Agra, 1981)
63. (a) Show that the Poisson bracket of two functions F and G does not obey commutative law but obeys distributive law of algebra.
- (b) If H is Hamiltonian and f is any function depending on position, momenta and time, show that $\frac{df}{dt} = \frac{\partial f}{\partial t} + \{H, f\}$ where $\{ \}$ stands for Poisson bracket.
(Rohilkhand 1989, 86, 84, 83)
64. What are Poisson Brackets ? If $\{u, v\}$ are the Poisson bracket, then prove that
$$\{u, v\} \omega = \{u, \omega\} v + \{v, \omega\} u$$

Show that the values of all Poisson brackets are independent of the sets of canonical coordinates they are expressed in.
(Agra 1998)
65. State the important properties of Poisson brackets. Show that Poisson bracket satisfies Jacobi Identity
$$[u, [v, w]] + [v, [w, u]] + [w, [u, v]] = 0$$
66. (a) Define Poisson bracket.
(Meerut 2009, 1991)
- (b) Show that the Poisson bracket of two integrals of motion is itself an integral of motion.
(Garhwal 1991)
- (c) Prove that the Poisson bracket must be invariant under canonical transformations.
(Meerut 2009, Rohilkhand 1991, Garhwal 1991)
67. Define the Poisson brackets and determine them for linear momentum $\vec{p}$ and angular momentum $\vec{L}$ of a particle.
(Rohilkhand 2000)
68. Define Lagrange and Poisson brackets and establish the relation.
$$\sum_{i=1}^{2n} \left\{ u_i, u_i \right\} \left[u_i, u_j \right] = \delta_{ij}$$

for a set of $2n$ independent functions u_i of coordinates $(q_1, \dots, q_n)$ $(p_1, \dots, p_n)$
(Garhwal 1999, Meerut 1997)
69. (a) Define Poisson and Lagrange brackets
- (b) Show that the Poisson bracket of any dynamical variable and a constant is equal to zero.
- (c) Explain the following equation
$$\frac{dA}{dt} = \frac{\partial A}{\partial t} + [A, H]$$

(Meerut 1998)
70. Define laboratory and centre of mass coordinate systems. Obtain the relation between angle of scattering in centre of mass and laboratory coordinate system.
(Rohilkhand 2001, 2000)
71. What is Hamilton principle ? Derive Hamilton's equations from Hamilton's principle.
(Rohilkhand 1986)

72. Use Hamilton's equations to find the differential equation for planetary motion and prove that the areal velocity remains constant. (Rohilkhand 1986 ; Meerut 1991)

73. (a) Define Poisson bracket.

- (b) Show that the Poisson bracket is invariant under a canonical transformation.

$$\{x, y\}_{q, p} = \{x, y\}_{Q, P}$$

- (c) Show that the Poisson bracket of any dynamical variables is invariant under a canonical transformation. Explain Hamilton equations of motion in Poisson bracket notations. (Meerut 2004)

74. What are Poisson brackets and their properties ? Discuss Poisson brackets of charged particle in an electromagnetic field. Obtain these brackets from the cartesian component of momentum $\mathbf{p}$ and angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ of a particle. (Rohilkhand 1998)

75. (a) Discuss Hamilton-Jacobi theory. In what circumstances is the characteristic function more useful than the special function?

- (b) Apply the Hamilton-Jacobi theory to one-dimensional Hamiltonian

$$H = \frac{1}{2}(p^2 + q^2) \text{ to deduce the motion of a particle. (Rohilkhand 1993, 85)}$$

76. Describe Hamiltonian-Jacobi theory and explain Hamilton's principal function. Use it to solve the problem of the linear harmonic oscillator.

(Rohilkhand 2001, 1993, Meerut 1992, 82, 79, Garhwal 1990)

77. Discuss the harmonic oscillator problem using (i) Hamilton-Jacobi method (ii) Canonical transformations. (Rohilkhand 1991)

78. Prove by Hamilton-Jacobi theory that the orbit of a planet round the sun is an elliptic one with the sun at its one of the foci. (Rohilkhand 1987, Meerut 1983, 80)

79. State and prove Hamilton-Jacobi equation for Hamilton's principal function and explain how it can be used to solve Kepler's problem for particle in an inverse square central force field. (Rohilkhand 1999, Agra 1981)

80. (a) Discuss the Hamilton-Jacobi theory for the case where energy is conserved.

- (b) A mass m is released from rest at a height h . Solve the equation of motion for this system using the Hamilton-Jacobi theory. (Rohilkhand 1990, 88)

81. (a) Obtain Hamilton-Jacobi equation for conservative and non-conservative systems.

- (b) What are Legendre transformations and the canonical variables ? (Rohilkhand 1998)

82. (a) Describe the method of action angle variable and calculate the frequency of linear harmonic oscillator using the method of action-angle variables.

- (b) Using Hamilton-Jacobi theory solve the Kepler problem for a particle in an inverse square central force field with the Hamiltonian

$$H = \frac{1}{2m} \left(p_r^2 + \frac{p_\theta^2}{r^2} \right) - \frac{k}{r} \quad (\text{Rohilkhand 1999})$$

83. (a) What are Euler's angles ? Find the transformation matrix in terms of these angles.

- (b) Obtain relation connecting the components of angular velocity along body axes to Euler's angles. (Meerut 2009)

84. Deduce Euler's equations of motion and apply them to solve the force free motion of a symmetrical rigid body. (Meerut 1992, 91, 90)

85. (a) A rigid body is rotating about an axis through the origin. Obtain relations connecting the components of total angular momentum and the components of angular velocity.

- (b) Obtain an expression for the rotational kinetic energy of a rigid body. (Meerut 1986)

86. If $A = B = C$, prove that the kinetic energy of rotation of rigid body referred to principal axis is given in terms of Eulerian angles as

$$T = \frac{1}{2} A [\dot{\psi}^2 + \dot{\theta}^2 + 2 \dot{\phi} \dot{\psi} \cos \theta]$$

87. Solve the problem of a compound pendulum by using Euler's equation.
88. A plane is fixed in space. Co-ordinate axes OX, OY, OZ rotate about O . Their angular velocity has components $\omega_x, \omega_y, \omega_z$ along them. If the equation of the plane at any instant is $Ax + By + Cz = 1$, prove that

$$\frac{dA}{dt} = B\omega_y - C\omega_z, \frac{dB}{dt} = C\omega_x - A\omega_z, \frac{dC}{dt} = A\omega_y - B\omega_x.$$

89. Prove that a top cannot move in steady precession with spin s and inclination θ to the vertical unless $C^2 s^2 \geq 4Amga \cos \theta$.
90. Define Euler's angles and obtain an expression for the complete transformation matrix. (Meerut 1983)
91. Show that the angular momentum $\mathbf{J}$ of a rigid body is given by

$$\mathbf{J} = \mathbf{I} \vec{\omega}$$

where $\vec{\omega}$ is angular velocity. Show that the operator I is a tensor of second rank.

92. (a) Obtain Euler's equations of motion for a rotating rigid body. (Meerut 1982)
(b) Obtain the condition that a heavy symmetrical top in the gravitational field which starts spinning initially with its symmetry axis vertical may continue to spin in the same way for an indefinite time. (Meerut 1967)
93. Discuss Euler's angles as the generalised coordinates for a rigid body motion. Obtain an expression for the angular velocity of a rigid body in terms of Euler's angles. (Meerut 2004, 2002, 1980)
94. Prove Euler's theorem for the motion of a rigid body with one point fixed. (Meerut 1997, 96)
95. Discuss stable and unstable equilibrium for a system executing small oscillations. Obtain Lagrange equation of motion for small oscillations. (Kerala 2002)
96. (a) Show that the components of the angular velocity of a rigid body, along the space set of axes are given in terms of Euler angles θ, ϕ, ψ by

$$\omega_x = \cos \phi \dot{\theta} + \sin \theta \sin \phi \dot{\psi}$$

$$\omega_y = \sin \phi \dot{\theta} - \sin \theta \cos \phi \dot{\psi}$$

$$\omega_z = \cos \theta \dot{\psi} + \dot{\phi}$$

- (b) Discuss torque-free motion of a symmetrical rigid body. (Meerut 1998)
97. (a) Obtain the Hamilton's-Jacobi equation for the Hamilton's principal function and prove that the Hamilton's principal function is a generator of a canonical transformation to construct coordinates and momenta. What is the Hamilton's characteristic function?
(b) Discuss the conservation of angular momentum and the associated symmetry operation. (Rohilkhand 2001)
98. Describe the method of canonical transformation of dynamical variables such that the dynamical equations remain invariant. (Kerala 2003)
99. Show by the invariance of the Poisson bracket, that the transformation from q, p to Q, P by the equations

$$Q_1 = q_1^2 + p_1^2, Q_2 = \frac{1}{2}(q_1^2 + q_2^2 + p_1^2 + p_2^2)$$

$$P_1 = \frac{1}{2} \left(-\tan^{-1} \frac{q}{p_1} + \tan^{-1} \frac{q_2}{p_2} \right); P_2 = -\tan^{-1} \frac{q_2}{p_2}$$

is canonical.

(Kerala 2003)

100. Show that the trajectory of a particle of mass m in a central potential $V(r)$ can be written as

$$\dot{r} = \sqrt{\frac{2}{m} \left(E - V + \frac{J^2}{2mr^2} \right)}$$

where E is the energy and J is angular momentum.

Discuss the nature of orbits as a function of $\left[E - V + \frac{J^2}{2mr^2} \right]$.

(Kerala 2003)

101. Show that a Lagrangian of a system with many degrees of freedom for small oscillations about equilibrium can be written as

$$L = \frac{1}{2} (m_{jk} \dot{\eta}_j \dot{\eta}_k + k_{jk} \eta_j \eta_k)$$

where η 's are small displacements and m_{jk}, k_{jk} are real symmetric constants. Find the form of the solutions η .

(Kerala 2003)

102. For the Lagrangian

$$L = \sum_i f_i(q_i) \dot{q}_i^2 - \sum_i V_i(q_i)$$

obtain the Hamiltonian and the Hamilton's equations of motion.

(Kerala 2003)

103. The Lagrangian of a particle moving in a central force is

$$L = \frac{1}{2} m \dot{r}^2 + \frac{p_\theta^2}{2mr^2} - V(r)$$

Find the equation of motion.

104. Show that $[p_i, f(p_i, q_i)]_{Q, P} = -\frac{\partial f}{\partial q_i}$, where bracket denotes Poisson bracket and q_i, p_i are function of Q_i and P_i

(Kerala 2002)

105. What do you mean by normal mode and eigen frequencies? Discuss double pendulum in the light of theory of small oscillations.

(Meerut 2004)

106. What do you mean by stable and unstable equilibrium? Establish the Lagrangian and deduce the Lagrange's equations of motion for small oscillations of a system in the neighbourhood of stable equilibrium.

(Meerut 2002)

107. Write short notes on:

1. D' Alembert's Principle.

(Rohilkhand 1989)

2. Atwood' Machine

(Rohilkhand 1989)

3. Interchangeability between orders of variation and differentiation

(Rohilkhand 1989)

4. Routh's Procedure.

(Rohilkhand 1989)

5. Hamilton-Jacobi equation

(Rohilkhand 1989)

6. Brachisto-chrome problem.

(Meerut 1986, Rohilkhand 1988)

7. Hamilton's variational principle.

(Rohilkhand 1988)

8. Generalised force.

(Rohilkhand 1988)

9. Symmetry properties and conservation theorems

(Rohilkhand 1992)

10. Generalised momenta

(Rohilkhand 1994)

11. Rutherford scattering of α -particles

(Rohilkhand 2000)

12. Kepler Problem

(Rohilkhand 2000)

108. Explain non-linear motion in one dimension with an example.

109. State pendulum equation in linear and non-linear forms. Find the solution of equation in each case.

110. State the equation of linear harmonic oscillator. Find its solution by phase-trajectory method.

111. Discuss the motion of a damped harmonic oscillator in the following cases :
 (i) underdamped motion, (ii) overdamped motion.
 What is the form of phase trajectory in each case ?
112. What are singular points ? Define and tell the type of following singular points.
 (i) vortex point, (ii) saddle point, (iii) focal point, (iv) nodal point.
 Also mention the type of equilibrium about above points.
113. What do you mean by limit cycle ? Explain taking the example of Vander Pol Oscillator.
114. Draw a simple non-linear electrical circuit using a junction diode. What do you mean by period doubling ? Explain the occurrence of chaos in the circuit.
115. Illustrate the essential features of chaos using logistic map. (Calicut 2002)
116. (a) Write down the non-linear differential equation of the pendulum and show that the first integral of the equation is the mechanical energy of system.
 (b) Draw the phase portrait of the above pendulum and use it to describe its motion. (Calicut 2003)
117. (a) Explain what is meant by logistic map, taking the difference equation
 $x_{n+1} = \mu x_n (1 - x_n)$. x_n in the range (0, 1) and $1 < \mu < 3$ as an illustrative example.
 (b) What is bifurcation (period doubling) in chaotic system. Draw the bifurcation diagram for the above case and explain. (Calicut 2002)
118. (a) What do you mean by invariant tori ?
 (b) State and explain KAM theorem.

Objective Questions : Select the right Choice

- The generalised coordinates are
 (a) dependent (b) independent
 (c) spherical polar coordinates (d) definitely not cartesian coordinates.
- A particle is constrained to move along the inner surface of a fixed hemispherical bowl, the number of degrees of freedom of the particle is (Agra 2007)
 (a) one (b) two (c) three (d) four.
- the constraint involved on a bead on a uniformly wire in a force free space is
 (a) Rheonomic (b) Scleronomic
 (c) both (a) and (b) (d) neither (a) nor (b). (Agra 2007)
- A particle is moving along the arc of a circle. The suitable generalised coordinates is/are
 (a) r only (b) θ only
 (c) r and θ both (d) x and y . (Agra 2008)
- Constraint in a rigid body is (Agra 2010)
 (a) halonomic (b) non-holonomic
 (c) scleronomic (d) none of these.
- For a conservative system the potential energy does not depend upon
 (a) force (b) generalised coordinate
 (c) generalised velocity (d) all of the above.
- The total virtual work done by effective force in any dynamical system compatible with constraints is zero. This is called. (Agra 2008)
 (a) D'Alemberts' principle (b) Principle of virtual norm
 (c) Lagrange's principle (d) Hamilton's principle.

8. D'Alemberts' principle is
- (a) $\sum_i (\mathbf{F}_i - \dot{\mathbf{p}}_i) \cdot \delta \mathbf{r}_i = 0$ (b) $\sum_i (\mathbf{F}_i - \dot{\mathbf{p}}_i) \cdot \delta \mathbf{r}_i = 0$
 (c) $\sum_i \mathbf{F}_i \cdot \delta \mathbf{r}_i = 0$ (d) $\sum_i (\mathbf{F}_i + \dot{\mathbf{p}}_i) \cdot \delta \mathbf{r}_i = 0$
9. The Lagrangian of a system is given by
- (a) $L = T - V$ (b) $L = T + V$ (c) $L = 2T + V$ (d) $L = 2T - V$
10. The Lagrangian for a simple pendulum (assuming equilibrium position of bob as reference level for zero potential energy) is
- (a) $\frac{1}{2} m l^2 \dot{\theta}^2 - m g l \cos \theta$ (b) $\frac{1}{2} m l^2 \dot{\theta}^2 + m g l \cos \theta$
 (c) $\frac{1}{2} m l^2 \dot{\theta}^2 - m g l (1 - \cos \theta)$ (d) $\frac{1}{2} m l^2 \dot{\theta}^2 + m g l (1 - \cos \theta)$
11. The Lagrange's equation for LC circuit is
- (a) $L \frac{d^2 q}{dt^2} - \frac{q}{C} = 0$ (b) $L \frac{d^2 q}{dt^2} - \frac{C}{q} = 0$ (c) $LC \frac{d^2 q}{dt^2} + \frac{q}{C} = 0$ (d) $L \frac{d^2 q}{dt^2} + \frac{q}{C} = 0$
 (where q is charge, L is inductance, C is capacitance)
12. A coordinate which does not appear in the Lagrangian is called
- (a) cyclic (b) free (c) generalised (d) holonomic.
13. Mutual interaction forces between two particles can change. (Agra 2010)
- (a) the linear momentum but not the kinetic energy.
 (b) the kinetic energy but not the linear momentum.
 (c) the linear momentum as well as kinetic energy.
 (d) neither linear momentum nor kinetic energy.
14. If a coordinate does not appear in the Lagrangian, then the corresponding conserved quantity is
- (a) linear momentum (b) generalised momentum
 (c) angular momentum (d) mechanical energy.
15. If L is Lagrangian of a system and q_k is generalised coordinate, then generalised momentum is (Agra 2008)
- (a) $\frac{\partial L}{\partial q_k}$ (b) $\frac{\partial L}{\partial \dot{q}_k}$ (c) $\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k}$ (d) $\frac{\partial L}{\partial \dot{q}_k} + \frac{\partial L}{\partial q_k}$
16. If the Lagrangian does not depend on time explicitly, then the corresponding conserved quantity is
- (a) linear momentum (b) generalised momentum
 (c) angular momentum (d) mechanical energy.
17. A system of particles contains n interacting particles and its equations of constraints are represented by l -equations, then the number of generalised coordinates is
- (a) $(n - l)$ (b) $3n$ (c) $(3n - l)$ (d) $3(n - l)$
18. A rigid body moving freely in space has degrees of freedom. (Rohilkhand 2003)
- (a) 3 (b) 4 (c) 6 (d) 9.
19. Phase space is
- (a) momentum space
 (b) configuration space
 (c) superposition of position and momentum space
 (d) four-dimensional Minkowskian space.
20. Hamiltonian is the function of
- (a) generalised coordinates and generated velocities
 (b) generalised coordinates and generalised momenta

(Agra 2009)

- (c) generalised velocities and generalised momenta
(d) constants.

21. The Hamiltonian is defined as ($L = \text{Lagrangian}$)

$$\begin{array}{ll} (a) H = \sum_k p_k \dot{q}_k + L & (b) \sum_k p_k \dot{q}_k - L \\ (c) \sum_k q_k \dot{p}_k + L & (d) \sum_k q_k \dot{p}_k - L \end{array}$$

22. For a conservative system, Hamiltonian is

$$(a) H = T + V \quad (b) H = T - V \quad (c) H = 2T - V \quad (d) H = 2T + V$$

23. Hamilton's equations are

$$\begin{array}{ll} (a) \dot{q}_k = -\frac{\partial H}{\partial p_k}, \quad \dot{p}_k = \frac{\partial H}{\partial q_k} & (b) \dot{q}_k = \frac{\partial H}{\partial p_k}, \quad \dot{p}_k = -\frac{\partial H}{\partial q_k} \\ (c) \dot{q}_k = \frac{\partial H}{\partial p_k}, \quad \dot{p}_k = \frac{\partial H}{\partial q_k} & (d) \dot{q}_k = \frac{\partial H}{\partial \dot{p}_k}, \quad \dot{p}_k = -\frac{\partial H}{\partial \dot{q}_k} \end{array}$$

24. If a system is homogenous in space, it follows

- (a) conservation of mass (b) conservation of linear momentum
(c) conservation of angular momentum (d) conservation of energy.

25. If a system is isotropic in space, it follows

- (a) conservation of mass (b) conservation of linear momentum
(c) conservation of angular momentum (d) conservation of energy.

26. If a system is homogenous in time, it follows :

- (a) conservation of mass (b) conservation of linear momentum
(c) conservation of angular momentum (d) conservation of energy.

27. The Hamiltonian of a simple pendulum (taking equilibrium position of bob as reference level for zero potential energy) is

$$\begin{array}{ll} (a) H = \frac{p_\theta^2}{2ml^2} - mgl(1 - \cos \theta) & (b) H = \frac{p_\theta^2}{2ml^2} + mgl(1 - \cos \theta) \\ (c) H = \frac{p_\theta^2}{2ml^2} - mgl & (d) H = \frac{p_\theta^2}{2ml^2} + mgl \end{array}$$

28. The Hamiltonian corresponding to the Lagrangian $L = a\dot{x}^2 + b\dot{y}^2 - kxy$ is

$$\begin{array}{ll} (a) \frac{b_x^2}{2a} + \frac{b_y^2}{2b} - kxy & (b) \frac{b_x^2}{2a} + \frac{b_y^2}{2b} + kxy \\ (c) \frac{b_x^2 + b_y^2}{4ab} + kxy & (d) \frac{b_x^2}{4a} + \frac{b_y^2}{4b} + kxy \end{array}$$

29. If the Lagrangian does not depend on the explicitly, then

(Agra 2010, 2007)

- (a) Hamiltonian is constant (b) Hamiltonian cannot be constant
(c) Potential energy is constant (d) Kinetic energy is constant.

30. The reduced mass of two bodies of masses m_1 and m_2 is ($\mu =$

$$(a) \frac{m_1 + m_2}{2} \quad (b) \sqrt{m_1 m_2} \quad (c) \frac{m_1 m_2}{m_1 + m_2} \quad (d) \frac{(m_1 + m_2)^3}{m_1 m_2}$$

31. A massless spring of force constant κ has masses m_1 and m_2 , attached at its two ends, the system rests on a horizontal table. The angular vibrational frequency ' ω ' of this system is

$$\begin{array}{ll} (a) \left(\frac{\kappa}{m_1 - m_2} \right)^{1/2} & (b) \left(\frac{\kappa}{m_1 + m_2} \right)^{1/2} \\ (c) \left[\frac{\kappa (m_1 + m_2)}{m_1 m_2} \right]^{1/2} & (d) \left[\kappa \left(\frac{1}{m_1} - \frac{1}{m_2} \right) \right]^{1/2} \end{array}$$

32. A physical system is invariant under rotation about a fixed axis. Then which of the following quantity is conserved :
- Total linear momentum
 - Linear momentum along the axis of rotation
 - Total angular momentum
 - Angular momentum along the axis of rotation.
33. A particle on a rotating parabola can be described by the Lagrangian

$$L = \frac{1}{2} [m \dot{x}^2 - \kappa x^2 + \lambda x^2 \dot{\theta}^2], \quad \dot{x} = \frac{dx}{dt}$$

where λ and κ are constants, then the canonically conjugate momentum p is

- $m \dot{x}$
- $m (1 + \lambda^2) \dot{x}$
- $m \lambda x^2 \dot{x}$
- $m (1 + \lambda x^2)^{-1} \dot{x}$

34. Euler-Lagrange's equation for function $f(y, y', x)$; (where $y' = \frac{dy}{dx}$) is

- $\frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y} = C$
- $\frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) - \left(\frac{\partial f}{\partial y} \right) = 0$
- $\frac{d}{dx} \left(\frac{\partial f}{\partial y} \right) - \frac{\partial f}{\partial y'} = 0$
- $\frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) + \frac{df}{dx} = 0.$

35. Hamilton's principle is

- $\int_{t_1}^{t_2} L dt = 0$
- $\delta \int_{t_1}^{t_2} L dt = 0$
- $\int_{t_1}^{t_2} \left(\sum_k p_k \dot{q}_k - L \right) dt = 0$
- $\delta \int_{x_1}^{x_2} \left(\sum_k p_k \dot{q}_k - L \right) dx = 0.$

36. Modified Hamilton's principle is

(Agra 2009)

- $\delta \int_{t_1}^{t_2} L dt = 0$
- $\delta \int_{t_1}^{t_2} H dt = 0$
- $\delta \int_{t_1}^{t_2} \left(\sum_k p_k \dot{q}_k - H \right) dt = 0$
- $\Delta \int_{x_1}^{x_2} \left(\sum_k p_k \dot{q}_k - L \right) dt = 0.$

37. The principle of least action is

- $\Delta \int_{t_1}^{t_2} L dt = 0$
- $\Delta \int_{t_1}^{t_2} p_k \dot{q}_k dt = 0$
- $\delta \int_{t_1}^{t_2} H dt = 0$
- $\delta \int_{t_1}^{t_2} q_k \dot{p}_k dt = 0.$

38. In δ -variation

- time t is fixed at end points only
- time t is fixed at all positions of path
- time t varies at all positions of path including end points
- time t is varied at end points only.

39. In Δ -variation
- time t and position coordinates are fixed at end points only
 - time t is variable at all positions of path including end points but position coordinates are fixed at end points
 - time t varies at end points only
 - time t varies but position is fixed at all positions of path.
40. The first integrals of motion of system under central force are
- linear momentum and energy
 - linear and angular momenta
 - angular momentum and energy
 - linear momentum, angular momentum and energy.
41. The areal velocity of a planet in terms of angular momentum (L) is given by (μ = reduced mass)
- $\frac{L}{\mu}$
 - $\frac{L}{2\mu}$
 - $\frac{2L}{\mu}$
 - $\sqrt{2\mu L}$.
42. If the energy of a planet were positive, the path of planet would be
- circular
 - elliptical
 - parabolic
 - hyperbola.
43. The orbit of alpha particles in Rutherford scattering by heavy nucleus is
- circular
 - elliptical
 - parabolic
 - hyperbola.
44. A particle describes a circular orbit under the influence of an attractive central force directed towards a point on the circle, then force varies as r^n , where n must be
- 2
 - 2
 - 4
 - 5.
45. A transformation from set (q, p) to (Q, P) is canonical if
- Hamilton's equations retain their form
 - Lagrange's equations retain their form
 - both Lagrange's and Hamilton's equations retain their form
 - Neither Lagrange's nor Hamilton's equations retain their form.
46. Rutherford differential scattering cross-section (Agra 2010)
- has the dimensions of area
 - has the dimensions of solid angle
 - is proportional to the square of the kinetic energy of incident particles
 - is inversely proportional to $\text{cosec}^4(d/x)$ where d is the angle of scattering.
47. The condition for transformation from (q, p) to (Q, P) is canonical if
- $P = p, Q = q$ only
 - $PQ = pq$
 - $PdQ - pdq$ is an exact differential
 - $PdQ + pdq$ is an exact differential.
48. If L and K are Lagrangian in old and new variables, then the transformation is canonical if
- $L + K = \frac{dF}{dt}$
 - $L - K = \frac{dF}{dt}$
 - $\frac{L}{K} = \frac{dF}{dt}$
 - $LK = \frac{dF}{dt}$.
49. Which one of the following transformations is canonical ?
- $P = q, Q = p$
 - $P = Q, Q = -P$
 - $Q = -p, P = -p$
 - $Q = p, P = -q$.
50. The Poisson bracket of two functions u and v with respect to coordinates (q, p) is
- $[u, v]_{q,p} = \frac{\partial u}{\partial q_k} \frac{\partial v}{\partial q_k} + \frac{\partial u}{\partial p_k} \frac{\partial v}{\partial q_k}$
 - $[u, v]_{q,p} = \frac{\partial u}{\partial q_k} \frac{\partial v}{\partial p_k} - \frac{\partial u}{\partial p_k} \frac{\partial v}{\partial q_k}$
 - $[u, v]_{q,p} = \frac{\partial q_k}{\partial u} \frac{\partial p_k}{\partial v} + \frac{\partial q_k}{\partial v} \frac{\partial p_k}{\partial u}$
 - $[u, v]_{q,p} = \frac{\partial q_k}{\partial u} \frac{\partial p_k}{\partial v} - \frac{\partial q_k}{\partial v} \frac{\partial p_k}{\partial u}$.
51. The Poisson bracket of two integrals of motion is
- zero
 - unity
 - infinite
 - integral of motion.
52. If a function f does not depend on time explicitly, then it is constant of motion if
- $[H, f] = 0$
 - $[H, f] = 1$
 - $[H, f] = -1$
 - $[H, f] = \text{non-zero finite}$.

53. Hamilton-Jacobi equation is
- (a) $\frac{\partial S}{\partial t} + H(q, Q, t) = 0$ (b) $\frac{\partial S}{\partial t} + H\left(q, \frac{\partial S}{\partial q}, t\right) = 0$
- (c) $\frac{\partial H}{\partial t} + S(q, Q, t) = 0$ (d) $\frac{\partial H}{\partial t} + S\left(q, \frac{\partial H}{\partial t}, t\right) = 0$
54. If the Poisson bracket of a function in the Hamiltonian vanishes. (Agra 2010)
- (a) the function depends on time (b) the function is constant of motion
- (c) the function is not a constant of motion
- (d) none of these.
55. Hamilton's canonical equations in terms of Poisson bracket are : (Agra 2009)
- (a) $\dot{q}_k = [q_k, H], \dot{p}_k = [p_k, H]$ (b) $\dot{q}_k = [p_k, H], \dot{p}_k = [q_k, H]$
- (c) $\dot{q}_k = [H_1, q_k], \dot{p}_k = [H, p_k]$ (d) $\dot{q}_k = [H, p_k], \dot{p}_k = [H, q_k]$
56. Hamilton's characteristic function is identified as
- (a) Action (b) Angle (c) Hamiltonian (d) Lagrangian (Agra 2009)
57. The relation between time period 'T' of a planet and semi-major axis 'a' of an elliptical orbit is (Agra 2008)
- (a) $T \propto a$ (b) $T \propto a^{1/2}$
- (c) $T \propto a^{3/2}$ (d) $T \propto a^3$
58. In terms of Lagrangian L, Hamilton's principal function is given as
- (a) $S = \int L dq$ (b) $S = \int L dt$ (c) $S = \int L dt$ (d) $S = \int H dq dp$
59. Moment of inertia tensor has
- (a) 3-independent components (b) 6-independent components
- (c) 9-independent components (d) 36-independent components.
60. For what value of Eulerian angle θ , the kinetic energy of rotation of a symmetrical top with $I_1 = I_2 = I_3 = I$ will be given by $T = \frac{1}{2} I (\dot{\theta}^2 + \dot{\phi}^2 + \dot{\psi}^2)$?
- (a) 0 (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{2}$ (d) π
61. The principal moment of inertia of an ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ for axis of rotation about z-axis is
- (a) $\frac{M}{5} (a^2 + b^2)$ (b) $\frac{M}{5} (a^2 + c^2)$ (c) $\frac{M}{5} (b^2 + c^2)$ (d) $\frac{M}{5} (a^2 + b^2 + c^2)$
62. For one-dimensional harmonic oscillator $U = \frac{1}{2} kx^2$, the equation of motion is
- (a) $\ddot{x} + \frac{k}{M} x = 0$ (b) $\ddot{x} - \frac{k}{M} x^2 = 0$ (c) $\ddot{x} + \frac{m}{k} x = 0$ (d) $\ddot{x} - \frac{m}{k} x^2 = 0$
63. For the Lagrangian $L = \frac{1}{2} \dot{q}^2 - q\dot{q} + q^2$, the momentum conjugate to coordinate q is
- (a) $\dot{q} + q$ (b) $\dot{q}$ (c) $\dot{q} - q$ (d) $\frac{1}{4} \dot{q}^2 - \frac{q^2}{2}$
64. The Hamiltonian is given by $H = \frac{1}{2} (xp^2 + 2ypq + zq^2)$; the value of $\dot{q}$ is
- (a) $\frac{xp}{2}$ (b) $xp + yq$ (c) $-(xp + yq)$ (d) $(y + z)q$
65. Which of the following equation is non-linear ?
- (a) $F = -kx$ (b) $\frac{d^2x}{dt^2} = \frac{b}{m} x$ (c) $\frac{d^2x}{dt^2} = -kx^2$ (d) $\frac{d^2\theta}{dt^2} = -\omega^2\theta$

66. In pendulum motion, the integral of motion is
 (a) velocity (b) amplitude (c) linear momentum (d) energy.
67. An autonomous system is one whose equations of motion
 (a) are linear
 (b) contain time explicitly
 (c) do not involve time and position explicitly
 (d) do not involve time explicitly.
68. The phase trajectory of a differential equation $\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$ for constant energy is
 (a) a circle (b) a parabola (c) an ellipse (d) a hyperbola.
69. The differential equation $\frac{d^2x}{dt^2} = \frac{k}{m}x$.
 (a) is linear and represents periodic motion
 (b) is non-linear and represents periodic motion
 (c) is linear and represents aperiodic motion
 (d) is non-linear and represents aperiodic motion.
70. The phase trajectory of equation $\frac{d^2x}{dt^2} = \frac{k}{m}x$ is
 (a) a circle (b) a parabola (c) an ellipse (d) a hyperbola.
71. The phase trajectory of a damped oscillator in underdamped case is
 (a) a circle (b) an ellipse (c) a hyperbola (d) a spiral.
72. The motion about a singular point is a periodic and unstable. The singular point is a
 (a) a saddle point (b) vortex point (c) focal point (d) nodal point.
73. The motion about a singular point is a periodic but stable. The singular point is a
 (a) saddle point (b) vortex point (c) focal point (d) nodal point.
74. The logistic map function is
 (a) $x_{n+1} = A x_n (1 + x_n)$ (b) $x_{n+1} = A (1 - x_n^2)$
 (c) $x_{n+1} = A x_n (1 - x_n)$ (d) $x_{n+1} = A (1 + x_n^2)$.
75. In logistic map function, for $A < 1$, the number of fixed points is
 (a) 0 (b) 1 (c) 2 (d) 3.

ANSWERS

1. (b), 2. (b), 3. (a), 4. (b), 5. (a), 6. (c), 7. (a), 8. (b), 9. (a), 10. (c),
 11. (d), 12. (a), 13. (b), 14. (b), 15. (b), 16. (d), 17. (c), 18. (c), 19. (c), 20. (b),
 21. (b), 22. (a), 23. (b), 24. (b), 25. (c), 26. (d), 27. (b), 28. (d), 29. (a), 30. (c),
 31. (c), 32. (d), 33. (a), 34. (b), 35. (b), 36. (c), 37. (b), 38. (a), 39. (b), 40. (c),
 41. (b), 42. (d), 43. (d), 44. (d), 45. (a), 46. (a), 47. (c), 48. (b), 49. (d), 50. (b),
 51. (d), 52. (a), 53. (b), 54. (b), 55. (a), 56. (a), 57. (c), 58. (b), 59. (b), 60. (c),
 61. (a), 62. (a), 63. (c), 64. (b), 65. (c), 66. (d), 67. (d), 68. (c), 69. (c), 70. (d),
 71. (d), 72. (a), 73. (d), 74. (c), 75. (b).

16

Special Theory of Relativity

16.1. Frames of reference

Once a person A asked another person B to lengthen or shorten a particular line LM without touching it. The person B drew two lines CD and EF , one on each side, adjacent and parallel to the given line LM (Fig. 16.1). The line CD was longer than LM and line EF was shorter than LM . Then he said that the line LM is shorter than CD and longer than EF .

Thus we see that the same line at the same time is shorter relative to one and longer relative to the other line. This example clearly indicates that *the length is relative*.

Similarly, if the person A is standing on one side of the river and other person B is standing across the river as shown in Fig. 16.2, a stationary boat inside the river will appear to the left of A and to the right of B . Thus we see that the same boat at the same instant of time is towards left relative to A and towards right relative to B . This clearly indicates that the *position is relative*.

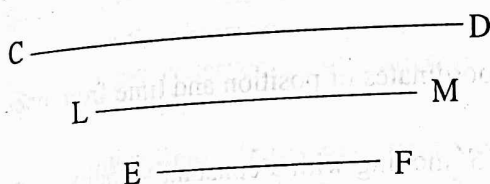


Fig. 16.1. Length is relative.

Now, let us suppose that a train moving with velocity 50 km./hour is observed by three observers, A , B and C ; A standing on the earth, B on a bicycle moving along the direction of propagation of train with velocity 10 km./hour, C sitting in a car moving with velocity 40 km./hour along the direction of propagation of the train. The person A would observe the velocity of train 50 km./hour, B would observe the velocity of train 40 km./hour and C would observe the velocity of the train 10 km./hour. Obviously the velocity of the same train is different relative to different observers. Thus we can say that the *motion is relative*.

Thus it is obvious that the motion of a body has no meaning unless it is described with respect to some well-defined coordinate system, known as *frame of reference* with respect to which the velocity of the body is measured. In any case a frame of reference must be chosen in such a way that the laws of nature may become fundamentally simpler when expressed in terms of such frames of reference. For this Newton introduced the idea of absolute space which was later on discarded.

Inertial and Non-Inertial Frames. There are generally two types of frames of reference:

- (i) Inertial frames
- (ii) Non-inertial or accelerated frames.

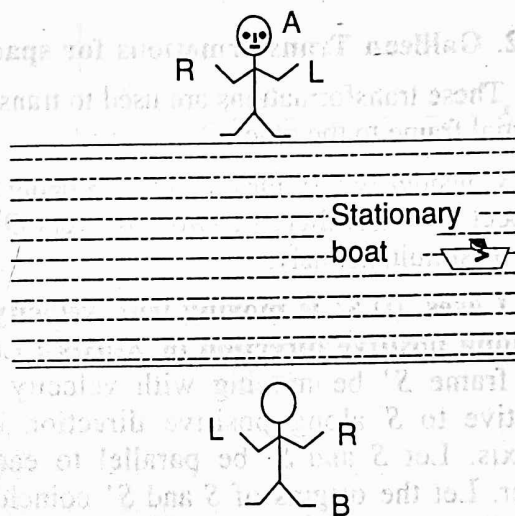


Fig. 16.2. Position is relative

Inertial frames: The frames relative to which an unaccelerated body appears to be unaccelerated, are called inertial frames.

Alternatively, the inertial frames may be defined as the frames in which Newton's laws are valid.

Non-inertial frames: The frames relative to which an unaccelerated body appears to be accelerated, are called non-inertial frames.

In these frames Newton's laws are not valid.

A frame of reference moving with constant velocity relative to an inertial frame is also inertial, since acceleration of body in both the frames is zero, while velocity of the body is different but uniform.

Experiments give an inference that a frame of reference fixed in stars is an inertial frame. A coordinate system fixed in earth is not an inertial frame, since earth rotates about its own axis and also revolves about the sun. In fact anything which is capable of turning is not an inertial frame, since apparent force, called the centrifugal force, acts on the body in the rotating frame.

To describe the motion of a body we should always choose inertial frames, for if we choose non-inertial frames, these involve different forces and hence it is very complicated to formulate the laws, common to all observers. Thus when mechanical laws are stated with reference to unaccelerated or inertial frames they preserve the same form whatever may be the relative motion between the observers, i.e., all frames of reference in uniform translatory motion relative to one another are equally suitable to express the laws of mechanics.

16.2. Galilean Transformations for space and time

These transformations are used to transform the coordinates of position and time from one inertial frame to the other.

Consider two frames S and S' , S being at rest and S' moving with a constant velocity with respect to S . Let there be two observers O and O' observing the event at any point P from S and S' simultaneously.

Cases. (i) S' is moving with velocity v along positive direction of X -axis : Let the frame S' be moving with velocity v relative to S along positive direction of X -axis. Let S and S' be parallel to each other. Let the origins of S and S' coincide initially.

Let the coordinates of P with respect to O and O' as origins be (x, y, z, t) and (x', y', z', t') .

If the event is happening at P at any particular time and the observations are taken by both the observers, from Fig. 16-3, we have

$$\left. \begin{aligned} x' &= x - vt, & y' &= y \\ z' &= z, & t' &= t \end{aligned} \right\} \quad \dots (1)$$

(ii) The frame S' is moving along a straight line relative to S along any direction : Let S' be moving relative to S with velocity $\vec{v}$ such that $\vec{v} = \hat{i} v_x + \hat{j} v_y + \hat{k} v_z$, where v_x, v_y and v_z are the components of velocity $\vec{v}$ along X, Y and Z axes respectively.

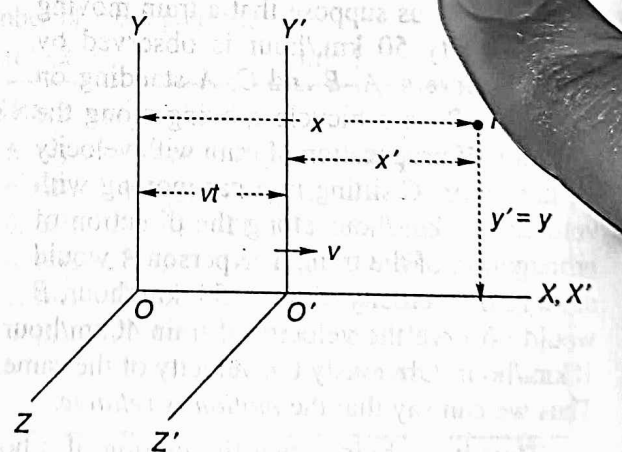


Fig. 16-3

If the origins of the two systems coincide initially and (x, y, z, t) and (x', y', z', t') are the coordinates of the event happening at P at any instant when the frame S' is separated from frame S by a distance $v_x t$, $v_y t$ and $v_z t$ along X , Y and Z axes respectively (Fig. 16-4), then we have

$$\left. \begin{aligned} x' &= x - v_x t, \quad y' = y - v_y t \\ z' &= z - v_z t, \quad t' = t \end{aligned} \right\} \dots(2)$$

The Galilean transformations of position in vector form may be represented by single equation by using vector addition law of triangle. If the origins of the two systems S and S' coincide initially and if $\mathbf{r}$ and $\mathbf{r}'$ are the position vectors of the event happening at P at any instant t when the frame S' has suffered a displacement $\mathbf{vt}$ relative to frame S , then from triangle $OO'P$ we have

$$\left. \begin{aligned} \mathbf{r} &= \mathbf{r}' + \mathbf{vt} \\ \text{or } \mathbf{r}' &= \mathbf{r} - \mathbf{vt} \\ \text{and } t' &= t \end{aligned} \right\} \dots(3)$$

The transformations represented by (1), (2) and (3) represent the Galilean transformations relating to the observations of position and time made by two observers in two different inertial frames.

(iii) **Galilean transformations for the velocity and acceleration of a particle :** Consider two systems S and S' , the latter moving with velocity $\mathbf{v}$ given by $\mathbf{v} = \hat{i} v_x + \hat{j} v_y + \hat{k} v_z$ relative to the former. If $\mathbf{r}$ and $\mathbf{r}'$ are the position vectors of any particle at time t as observed by observers in system S and S' respectively, we have from Galilean transformations,

$$\text{and } \left. \begin{aligned} \mathbf{r} &= \mathbf{r}' + \mathbf{vt} \\ t' &= t \end{aligned} \right\} \dots(1)$$

Differentiating above equations keeping $\mathbf{v}$ constant, we get

$$d\mathbf{r} = d\mathbf{r}' + \mathbf{v} dt \dots(2)$$

$$\text{Dividing, we get } \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}'}{dt} + \mathbf{v} \dots(3)$$

But $\frac{d\mathbf{r}}{dt}$ and $\frac{d\mathbf{r}'}{dt}$ are the velocities of the particle relative to systems S and S' respectively,

$$\text{i.e., } \frac{d\mathbf{r}}{dt} = \mathbf{u}, \quad \frac{d\mathbf{r}'}{dt} = \mathbf{u}'$$

Hence from (3), the Galilean, transformations for velocity may be expressed as

$$\mathbf{u}' = \mathbf{u} - \mathbf{v} \dots(4)$$

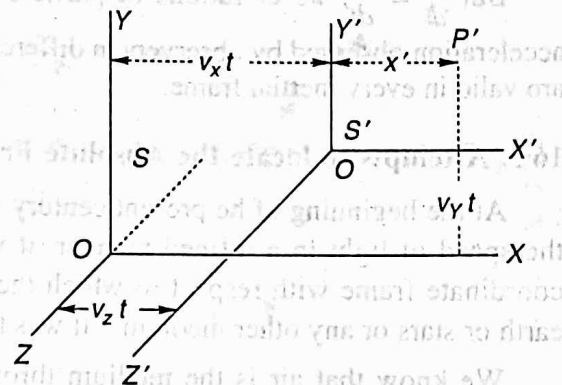


Fig. 16-4

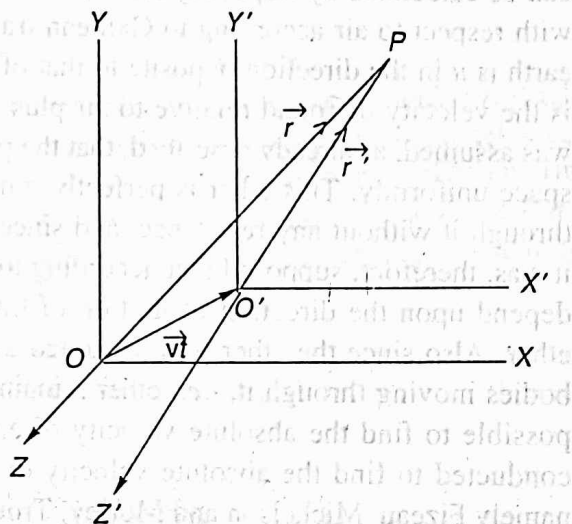


Fig. 16-5

Differentiating the above equation with respect to time t , we get

$$\frac{du'}{dt} = \frac{du}{dt} \quad \text{or} \quad \frac{du'}{dt'} = \frac{du}{dt} \quad (\text{since } dt' = dt)$$

But $\frac{du}{dt} = \frac{du'}{dt'}$ accelerations of particle in systems S and S' respectively, i.e., $a' = a$, i.e., acceleration observed by observers in different inertial frames is the same. Thus Newton's laws are valid in every inertial frame.

16.3. Attempts to locate the Absolute Frame : The speed of light relative to earth

At the beginning of the present century when the modern techniques were used to measure the speed of light in a refined manner, it was questioned what was the reference system or coordinate frame with respect to which the speed of light was measured? Was it relative to earth or stars or any other medium? It was to choose the medium for light as air in sound.

We know that air is the medium through which sound travels. Therefore, the speed of sound is measured with respect to air. We also know that the velocity of sound relative to earth can be calculated by imposing the velocity of air with respect to earth to the velocity of sound with respect to air according to Galilean transformations. If the velocity of air with respect to earth is u in the direction opposite to that of sound, then the velocity of sound relative to earth is the velocity of sound relative to air plus velocity of air relative to the earth, i.e., u . Then it was assumed, as already described, that the preferred medium for light was ether which filled all space uniformly. This ether is perfectly transparent to light and the material bodies may pass through it without any resistance, and since the earth has orbital motion about 3×10^4 m/sec, it was, therefore, supposed that according to Galilean transformations the speed of light should depend upon the direction of motion of light with respect to the earth's motion through the ether. Also since the ether was assumed to be at rest and it offered no resistance to material bodies moving through it, i.e., ether remains fixed in space, therefore, it was considered to be possible to find the absolute velocity of any body moving through it; but experiments were conducted to find the absolute velocity of the earth through ether by a number of scientists namely Fizeau, Michelson and Morley, Trouton and Nobel and many others; but they could not get success in their attempts. Later Hertz proposed that ether was not at rest; but it was carried completely by bodies moving through it, that is, the velocity of ether is the same as that of the body moving through it. By this assumption the phenomenon of *aberration* could not be explained and hence the assumption was rejected.

16.4. Michelson-Morley Experiment

According to Newtonian mechanics or classical mechanics there existed absolute luminiferous ether relative to which the measurements are taken. The ether is filled uniformly through all space and penetrates all matter and it was considered that light propagates through ether as the sound propagates through air.

The ether was assumed to be at rest. The light passes through ether with fixed velocity 3×10^8 m/s. and the earth passes through this stationary ether with velocity 3×10^4 m/s. Since the earth travels round the sun with a speed 3×10^4 m/s, therefore if a light signal is travelling in the direction of the motion of the earth, it should take longer time to travel the same distance than if it were sent in the opposite direction since in the former case the observer is moving away from the signal while in the latter case the observer is approaching the signal.

The experiment was used to measure the time difference between the two cases from which the relative velocity between the ether and the earth could be determined.

The apparatus used for measuring this time difference was Michelson's interferometer shown in Fig. 16-6.

It consists of a monochromatic light source S , semi-silvered plate P , inclined at 45° to the beam of light and two mirrors M_1 and M_2 , highly silvered on front surfaces to avoid multiple internal reflections. It also consists of plate Q such that P and Q are of equal thickness and of the same material, mounted parallel to each other. The plate Q does not reflect light at all.

The beam of light emitted from a monochromatic source S , falls upon the plate P . The plate P , due to being semi-silvered, divides the beam into two parts, namely, reflected and transmitted beams. The reflected beam moves towards mirror M_1 and at A it falls normally, hence it is reflected back to P and after being transmitted from P enters the telescope T . Now the transmitted beam moves towards mirror M_2 and falls normally at B after entering the plate Q , therefore, it is reflected by the mirror M_2 and retraces its path BP while at P it is reflected to enter the telescope. These two coherent beams entering the telescope give rise to the phenomenon of interference, thereby producing the interference fringes.

The beam reflected at P crosses the plate P twice while the other beam in the absence of Q lies wholly in air. Due to this the reflected beam at P has to travel an extra optical path $2(\mu - 1)t$, where μ is refractive index of the plate P and ' t ' is its thickness. This extra optical path is compensated by placing the plate Q in the way of transmitted wave. Thus the function of the plate Q is only to equalise the optical paths traversed by both the beams if mirrors M_1 and M_2 are at equal distances from P .

If the apparatus is at rest in ether, the two waves, reflected and transmitted through the glass plate P would take equal times to return to P . But actually the whole apparatus is moving with earth. Consider that direction of motion of the earth in the direction of the initial beam of light. due to motion of apparatus with earth, the optical path traversed by both the rays are not

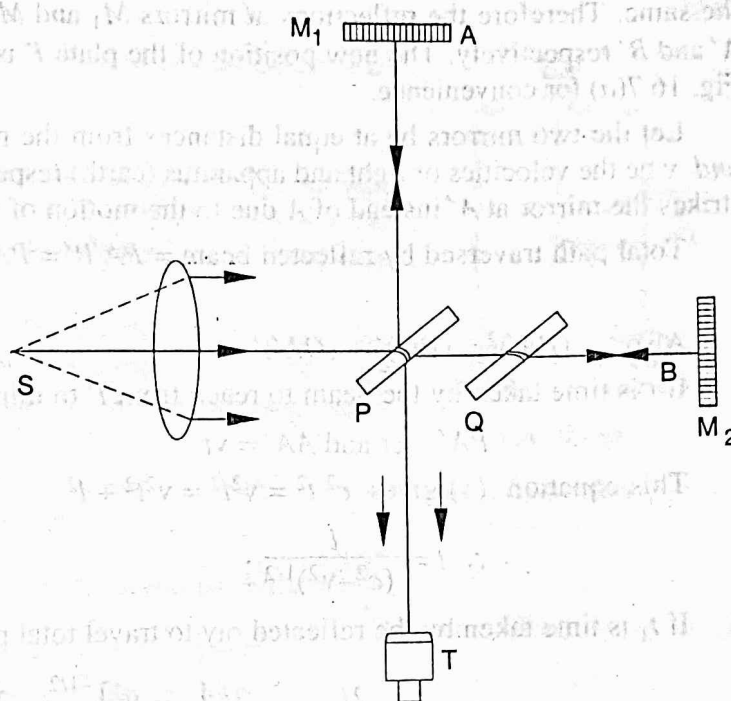


Fig. 16-6

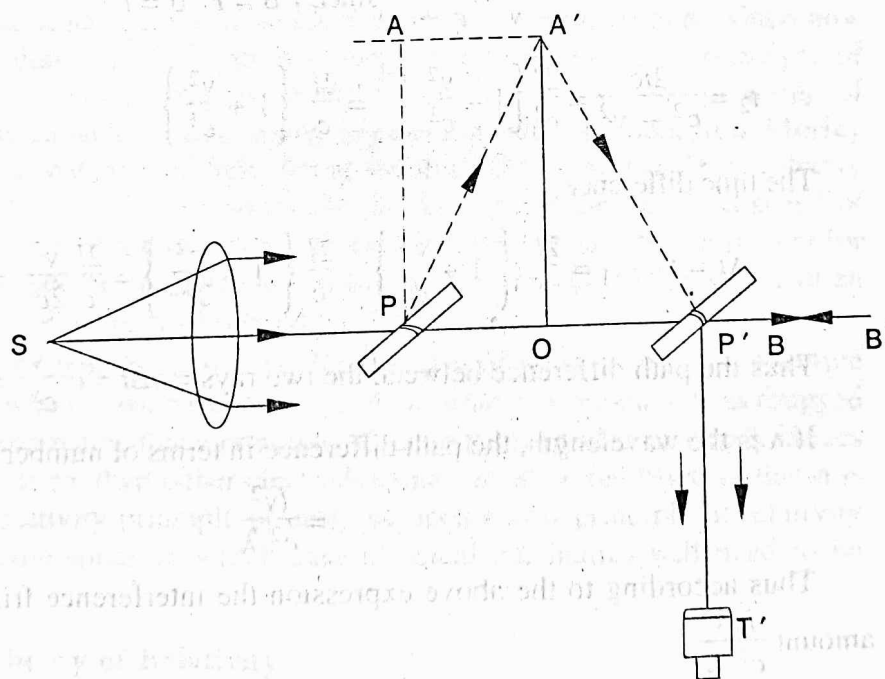


Fig. 16-7(a)

the same. Therefore the reflections at mirrors M_1 and M_2 do not take place at A and B , but at A' and B' respectively. The new position of the plate P is at P' and the plate Q is not shown in Fig. 16-7(a) for convenience.

Let the two mirrors be at equal distances from the plate, P , i.e., $PA = PB = l$ (say) and c and v be the velocities of light and apparatus (earth) respectively. Now, the ray reflected from P strikes the mirror at A' instead of A due to the motion of the earth.

Total path traversed by reflected beam = $PA'P' = PA' + A'P' = 2PA'$.

since $PA' = P'A'$

$$\text{Also } (PA')^2 = (PO)^2 + (OA')^2$$

If t is time taken by the beam to reach from P to mirror M_1 , then

$$PA' = ct \text{ and } AA' = vt$$

This equation (1) gives, $c^2 t^2 = v^2 t^2 + l^2$

since $PA = OP'$

$$\therefore t = \frac{l}{(c^2 - v^2)^{1/2}}$$

If t_1 is time taken by the reflected ray to travel total path, then

$$t_1 = 2t = \frac{2l}{(c^2 - v^2)^{1/2}} = \frac{2l}{c} \left\{ 1 - \frac{v^2}{c^2} \right\}^{-1/2} = \frac{2l}{c} \left\{ 1 + \frac{v^2}{2c^2} \right\} \quad \dots(2)$$

The transmitted ray has velocity $(c-v)$ relative to apparatus from P to B and $(c+v)$ from B' to P' ; since from P to B the transmitted ray and the apparatus are moving in the same direction while from B' to P' they are moving in opposite directions.

$\therefore$ Total time taken by transmitted beam to travel the total path, i.e., from P to B' and then from B' to $P' = t_2$,

$$= \frac{l}{c-v} + \frac{l}{c+v} \text{ since } PB = P'B' = l$$

$$\therefore t_2 = \frac{2lc}{c^2 - v^2} = \frac{2l}{c^2} \left[1 - \frac{v^2}{c^2} \right]^{-1} = \frac{2l}{c} \left\{ 1 + \frac{v^2}{c^2} \right\} \quad \dots(3)$$

The time difference

$$\Delta t = t_2 - t_1 = \frac{2l}{c} \left\{ 1 + \frac{v^2}{c^2} \right\} - \frac{2l}{c} \left\{ 1 + \frac{v^2}{2c^2} \right\} = \frac{2l}{c} \frac{v^2}{2c^2} = \frac{l}{c} \frac{v^2}{c^2} = \frac{lv^2}{c^3}$$

Thus the path difference between the two rays = $c\Delta t = \frac{lv^2}{c^2}$

If λ is the wavelength, the path difference in terms of number of fringes

$$= \frac{lv^2}{c^2 \lambda}$$

Thus according to the above expression the interference fringes should be shifted by amount $\frac{lv^2}{c^2 \lambda}$

In Michelson-Moreley experiment PA and PB were not exactly equal to get the straight fringes by proper inclination of the mirrors. The shift cannot be determined as such because earth is never at rest. In performing the experiment the transmitted beam travelled along direction of motion of apparatus. The whole apparatus was turned through 90° so that reflected and transmitted beams get interchanged. In this case the difference of path will be

opposite direction, the shift of fringes in two positions before and after 90° rotation is equivalent to $\frac{2l v^2}{c^2 \lambda}$

In spite of all the necessary precautions, the experiment showed no shift. The experiment was performed by a number of scientists at different times of year when the directions of earth's orbital velocity were different, but no shift was observed.

The negative results observed by the experiments suggest that the space or medium in which light propagates is not moving relative to earth. We may say that the *existence of stationary medium carrying light (or absolute space) is disproved.*

Ex. 1. Calculate the fringe-shift in Michelson-Morley experiment. Given that $l = 11$ metres, $v = 30$ km/sec, and $\lambda = 6 \times 10^{-5}$ cm.

Solution. The required fringe shift ΔN is given by $\Delta N = \frac{2l v^2}{c^2 \lambda}$

Here $l = 11$ metres, $\lambda = 6 \times 10^{-5}$ cm. $= 6 \times 10^{-7}$ metre

$v = 30$ km/sec. $= 3 \times 10^4$ m/sec; $c = 3 \times 10^8$ m/sec.

$$\therefore \Delta N = \frac{2 \times 11 \times (3 \times 10^4)^2}{(3 \times 10^8)^2 \times 6 \times 10^{-7}} = 0.37.$$

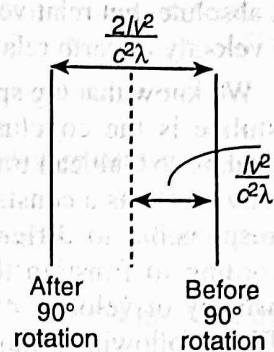


Fig. 16-7(b)

16.5. Explanation of negative results in Michelson-Morley Experiment : Principle of Constancy of Speed of Light

One way of explaining the negative results of Michelson-Morley experiment is to conclude simply that *the measured speed of light is the same (i.e., c) for all directions in every inertial frame.* This statement is called the *principle of constancy of the speed of light* and is one of the two fundamental postulates of Einstein's special theory of relativity. This principle would lead to $\Delta N = 0$, in Michelson-Morley experiment; since now the speed of light is c rather than $|c + v|$ in any frame, hence according to principle of constancy of speed of light no shift in fringes should be observed. Thus the principle of constancy of speed of light is capable of explaining negative results of Michelson-Morley experiment. The principle of constancy of light being incompatible with Galilean velocity transformations seemed to be too drastic philosophically at that time. If the observed speed of light did not depend on the motion of the observer, all inertial frames would be equivalent for propagation of light and there could be no experimental evidence to indicate the existence of an absolute ether frame. This led to the theory of relativity.

Thus there is no acceptable experimental basis for the idea of an ether, i.e., an absolute frame of reference, this is true whether we choose to regard the ether as stationary or as dragged along. We must now face the alternative that a principle of relativity is valid in electrodynamics as well as in mechanics. If this is so, then either electrodynamics must be modified so that it is consistent with the classical relativity principle or else, we need a new principle of relativity that is consistent with electrodynamics in which case classical mechanics will need to be modified.

16.6. Postulates of Special Theory of Relativity

(1) "The fundamental laws of physics have the same form for all inertial frames, i.e., for all reference frames at rest or moving with constant linear velocity relative to one another."

(2) The velocity of light in vacuum is independent of the relative motion of the source and the observer.

These are the two fundamental postulates used in the special theory of relativity. The first postulate is the extension of the conclusion drawn from Newtonian mechanics; since velocity is

not absolute, but relative which is a fact drawn from the failure of the experiments to determine the velocity of earth relative to ether.

We know that the speed of light is not constant under Galilean transformations and the first postulate is the conclusion from Newtonian mechanics; thus second postulate is not true according to Galilean transformations. Actually this is true since the velocity of light calculated by any means is a constant. Thus the second postulate is very important and only this postulate is responsible to differentiate the classical theory and Einstein's theory of relativity and according to Einstein the theory of relativity is applicable to laws of optics. Thus for the constancy of velocity of light we have to introduce the new transformation equations which fulfil the following requirements:

1. The speed of light c must have the same value in every inertial frame.
2. The transformation must be linear and for low speeds (i.e., $v \ll c$) they should approach the Galilean transformations.
3. They should not be based on "absolute time" and "absolute space".

16.7. Lorentz Transformation Equations

The above requirements were fulfilled by H.A. Lorentz by introducing transformation equations relating the observations of position and time made by two observers in two different inertial frames and are known as "Lorentz Transformation Equations".

Let S and S' be two inertial frames of reference, S' having uniform velocity v relative to S .

Let two observers O and O' observe any event P from systems S and S' respectively. For convenience let us consider that the X -axes of two systems coincide permanently and the velocity is parallel to X -axis. The event P is a light signal and is produced when both t and t' are zero and when origins of the two frames coincide. The event P is determined by coordinates (x, y, z, t) and (x', y', z', t') by observers O and O' respectively.

The light pulse produced at $t = 0$ will spread out as a growing sphere and the radius of wavefront produced in this way will grow with speed c . Since (x, y, z) are coordinates of the event relative to observer in system S at rest therefore the equation of spherical surface whose radius grows at the speed c , is

$$x^2 + y^2 + z^2 = c^2 t^2 \text{ or } x^2 + y^2 + z^2 - c^2 t^2 = 0 \quad \dots(1)$$

Similarly for observer O' in system S' having the coordinates of P as (x', y', z', t') , the equation of spherical surface is

$$x'^2 + y'^2 + z'^2 = c^2 t'^2 \text{ or } x'^2 + y'^2 + z'^2 - c^2 t'^2 = 0 \quad \dots(2)$$

c is not primed since according to II postulate of special theory it is a constant for all inertial frames.

Then from equations (1) and (2), we have

$$x^2 + y^2 + z^2 - c^2 t^2 = \lambda (x'^2 + y'^2 + z'^2 - c^2 t'^2) \quad \dots(3)$$

where λ is any undetermined constant.

As velocity of S' is only along x -axis; thus from symmetry

$$\left. \begin{aligned} y &= y' \\ z &= z' \end{aligned} \right\} \dots(4)$$

and

Therefore equation (3) gives $\lambda = 1$.

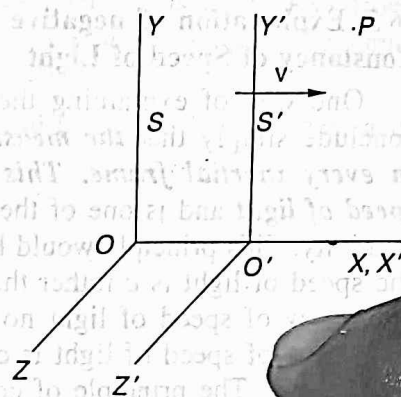


Fig. 16.8

Now for the transformation equation relating to x and x' , let us put

$$x' = \gamma(x - vt), \quad \dots(5)$$

γ being independent of x and t .

The reasons for trying the above relation are

(1) Transformation must reduce to Galilean transformation for low speed, i.e., for speed $v/c \rightarrow 0$.

(2) The transformation must be linear and simple. We see that above relation is sufficiently general to satisfy the required conditions.

Since motion is relative, we may assume that S is moving relative to S' with velocity $-v$ along (+) ve directions of x -axis, therefore

$$x = \gamma(x' + vt') \quad \dots(6)$$

Now putting the value of x' from (5) in (6), we have

$$x = \gamma[\gamma(x - vt) + vt']$$

$$\text{or} \quad t' = \gamma \left[t - \frac{x}{v} \left(1 - \frac{1}{\gamma^2} \right) \right] \quad \dots(7)$$

Now substituting in equation (3) the value of x' from equation (5), for t' from equation (7), $\lambda = 1$ and using (4) we have

$$x^2 - c^2 t^2 = \left[\gamma^2 (x - vt)^2 - c^2 \gamma^2 \left\{ t - \frac{x}{v} \left(1 - \frac{1}{\gamma^2} \right) \right\}^2 \right]$$

$$\text{or} \quad x^2 - c^2 t^2 - \gamma^2 (x^2 - 2vxt + v^2 t^2) + c^2 \gamma^2 \left\{ t - \frac{x}{v} \left(1 - \frac{1}{\gamma^2} \right) \right\}^2 = 0$$

$$\text{or} \quad x^2 - c^2 t^2 - \gamma^2 (x^2 - 2vxt + v^2 t^2) + c^2 \gamma^2 \left[t^2 - \frac{2xt}{v} \left(1 - \frac{1}{\gamma^2} \right) + \frac{x^2}{v^2} \left(1 - \frac{1}{\gamma^2} \right)^2 \right] = 0. \quad \dots(8)$$

Since this equation is an identity, the coefficient of various powers of x and t must vanish separately.

Equating coefficients of xt to zero, we have

$$\text{or} \quad 2\gamma^2 v - \frac{2c^2 \gamma^2}{v} \left(1 - \frac{1}{\gamma^2} \right) = 0 \quad \dots(9)$$

$$\text{or} \quad \gamma^2 (v^2 - c^2) + c^2 = 0. \quad \dots(10)$$

$$\text{i.e.,} \quad \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \quad \dots(11)$$

Further, from equation (5), we have $x' = \gamma(x - vt)$

From equation (7) we have $t' = \gamma \left[t - \frac{x}{v} \left(1 - \frac{1}{\gamma^2} \right) \right]$

$$= \gamma \left[t - \frac{x}{v} \left\{ 1 - \left(1 - \frac{v^2}{c^2} \right) \right\} \right] \text{ from equation (10)}$$

$$= \gamma \left[t - \frac{x}{v} \cdot \frac{v^2}{c^2} \right]$$

$$t' = \gamma \left[t - \frac{vx}{c^2} \right] \quad \dots(12)$$

Substituting value of γ from equation (10) in equations (11) and (12), we have

$$\left. \begin{aligned} x' &= \frac{x - vt}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)}} \\ t' &= \frac{t - \frac{vx}{c^2}}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)}} \end{aligned} \right\} \quad \dots(13)$$

Thus, combining (4) and (13), we have

$$x' = \frac{x - vt}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)}}, y' = y, z' = z \text{ and } t' = \frac{t - \frac{vx}{c^2}}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)}} \quad \dots(14)$$

These are called *Lorentz transformation equations*.

If we exchange our frames of reference or consider the given space time co-ordinates of the event to be those observed in system S' rather than in system S , the only change allowed by the relativity principle is the physical one of a change in relative velocity from v to $-v$, i.e., the system S moves relative to system S' with velocity v along negative direction of x -axis whereas system S' moves relative to system S along positive direction of x -axis. When we solve equation (14) for x, y, z and t in terms of the primed coordinates x', y', z' and t' we obtain.

$$x = \frac{x' + vt'}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)}}, y = y', z = z' \text{ and } t = \frac{t' + \frac{vx'}{c^2}}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)}}$$

These transformation equations are identical in form with equations (15) except that, as required, v changes to $-v$ and are *inverse Lorentz transformation equations*.

If $v \ll c$; then $v/c \ll 1$; hence Lorentz transformations (14) use the form

$$x' = x - vt, y' = y; z' = z \text{ and } t' = t \left(\text{since } \frac{vx}{c^2} \ll t \right) \quad \dots(16)$$

which are Galilean transformations. *Thus for low values of velocity v , the Lorentz transformations approach to Galilean.* Thus we say that Lorentz transformations hold for all velocities; while Galilean transformation hold only for low velocities. *Consequently Lorentz transformations are superior to Galilean transformations.*

Ex. 2. Show by direct application of Lorentz transformations $x^2 + y^2 + z^2 - c^2 t^2$ is invariant. (Agra 1998, Kanpur 1992)

Solution. We have only to prove by the help of Lorentz transformations

$$x^2 + y^2 + z^2 - c^2 t^2 = x'^2 + y'^2 + z'^2 - c^2 t'^2$$

where (x, y, z, t) and (x', y', z', t') are the coordinates of the same event observed by two observers in system S and S' while S' is moving with a velocity v relative to S .

Let us consider the expression $x'^2 + y'^2 + z'^2 - c^2 t'^2$

$$= \frac{(x - vt)^2}{1 - \frac{v^2}{c^2}} + y^2 + z^2 - c^2 \frac{\left(t - \frac{vx}{c^2}\right)^2}{1 - \frac{v^2}{c^2}}$$

[Putting values of x' , y' , z' and t' from Lorentz transformations].

$$\begin{aligned} x'^2 + y'^2 + z'^2 - c^2 t'^2 &= \frac{c^2}{c^2 - v^2} (x - vt)^2 + y^2 + z^2 - \frac{c^2}{c^2 - v^2} \cdot c^2 \left(t - \frac{vx}{c^2}\right)^2 \\ &= \frac{c^2}{c^2 - v^2} \left[(x - vt)^2 - c^2 \left(t - \frac{vx}{c^2}\right)^2 \right] + y^2 + z^2 \\ &= \frac{c^2}{c^2 - v^2} \left[x^2 - 2vxt + v^2 t^2 - c^2 \left(t^2 - \frac{2vxt}{c^2} + \frac{v^2 x^2}{c^4} \right) \right] + y^2 + z^2 \\ &= \frac{c^2}{c^2 - v^2} \left[x^2 - 2vxt + v^2 t^2 - c^2 t^2 + 2vxt - \frac{v^2 x^2}{c^2} \right] + y^2 + z^2 \\ &= \frac{c^2}{c^2 - v^2} \left[\frac{c^2 x^2 + c^2 v^2 t^2 - c^4 t^2 - v^2 x^2}{c^2} \right] + y^2 + z^2 \\ &= \frac{1}{c^2 - v^2} \left[c^2 (x^2 - c^2 t^2) - v^2 (x^2 - c^2 t^2) \right] + y^2 + z^2 \\ &= \frac{1}{c^2 - v^2} \left[(c^2 - v^2) (x^2 - c^2 t^2) \right] + y^2 + z^2 \\ &= x^2 - c^2 t^2 + y^2 + z^2 = x^2 + y^2 + z^2 - c^2 t^2 \end{aligned}$$

Thus we have proved that $x^2 + y^2 + z^2 - c^2 t^2 = x^2 + y^2 + z^2 - c^2 t^2$

That is, the expression $x^2 + y^2 + z^2 - c^2 t^2$ is invariant under Lorentz transformations.

16.8. Length Contraction

Consider two systems S and S' , the latter moving with velocity v relative to former along positive direction of X - axis.

Consider a rod parallel to x - axis placed in system S' at rest. Let x - co-ordinates of the ends of the rod in system S' be at x'_1 and x'_2 . Then length of the rod in system $S' = (x'_2 - x'_1) = L_0$ (say). i.e., $L_0 = x'_2 - x'_1$.

Let there be an observer O in system S . Obviously the observer O in S is moving with velocity v relative to the rod. Now we have to determine the length of the rod from system S .

Let x_1 and x_2 be the coordinates along x -axis of the ends of the rod at time t , t being the same for the both ends, i.e., the observations of both ends of rod are taken simultaneously.

Thus the length of the rod (at rest in S') observed by an observer from system S

$$L = (x_2 - x_1).$$

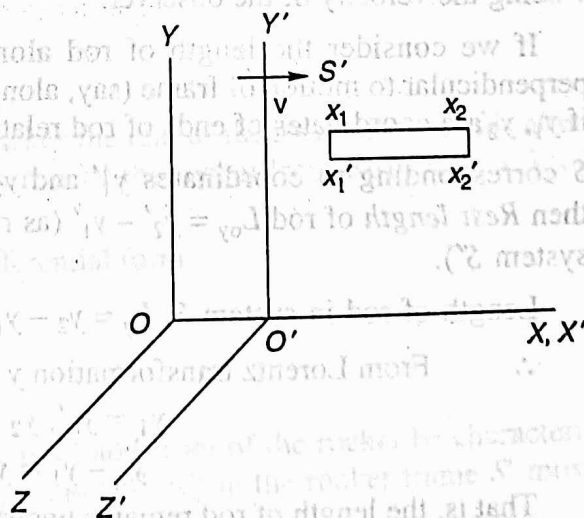


Fig. 16.9(a)

Now from Lorentz inverse transformation equations, we have

$$x = \frac{x' + vt'}{\sqrt{1 - \frac{v^2}{c^2}}} \quad \text{where } t' = \frac{t - (vx/c^2)}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\therefore x = \frac{x' + v \left(t - \frac{vx}{c^2} \right)}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$= \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} [x' + vt] - \frac{v^2}{c^2} \frac{x}{\left(1 - \frac{v^2}{c^2} \right)}$$

$$\therefore \left[1 + \frac{v^2/c^2}{(1 - v^2/c^2)} \right] x = \frac{1}{\sqrt{1 - v^2/c^2}} (x' + vt)$$

$$\therefore \frac{x}{1 - v^2/c^2} = \frac{x' + vt}{\sqrt{1 - v^2/c^2}}$$

$$\therefore x = (x' + vt) \sqrt{1 - v^2/c^2}$$

As both ends of the rod are being observed at same instant t , we have

$$x_1 = (x'_1 + vt) \sqrt{1 - (v^2/c^2)}; \quad x_2 = (x'_2 + vt) \sqrt{1 - (v^2/c^2)}$$

$$\text{Therefore } x_2 - x_1 = (x'_2 - x'_1) \sqrt{1 - \frac{v^2}{c^2}}$$

or

$$L = L_0 \sqrt{1 - \frac{v^2}{c^2}}$$

This shows that $L < L_0$.

This follows that the length of rod seems to be contracted when measured by moving observer along the direction of length of rod in the ratio $[1 : \sqrt{1 - v^2/c^2}]$, v being the velocity of the observer.

If we consider the length of rod along a direction perpendicular to motion of frame (say, along Y -axis) and if y_1, y_2 are coordinates of ends of rod relative to systems S corresponding to coordinates y'_1 and y'_2 in frame S' , then Rest length of rod $L_{oy} = y'_2 - y'_1$ (as rod is at rest in system S').

Length of rod in system S , $L_y = y_2 - y_1$.

$\therefore$ From Lorentz transformation $y = y'$, we have

$$y_1 = y'_1, \quad y_2 = y'_2$$

$$y_2 - y_1 = y'_2 - y'_1 \text{ i.e., } L_y = L_{oy}$$

That is, the length of rod remains unchanged in a direction perpendicular to the direction of motion.

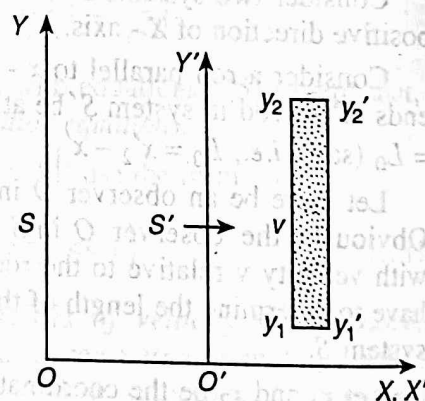


Fig. 16.9(b)

The length of moving rod is contracted along the direction of motion by a factor $\sqrt{1-v^2/c^2}$; while there is no contraction along a direction perpendicular to the direction of motion.

This statement is called **Lorentz-Fitzgerald contraction**.

SOLVED EXAMPLES

Ex. 3. A rod has length 1 metre, When the rod is in a satellite moving with velocity $0.8c$ relative to laboratory, what is the length of the rod as determined by an observer (a) in the satellite and (b) in the laboratory?

Solution. (a) The observer in the satellite is at rest relative to the rod, therefore the length of the rod as measured by an observer in the satellite is 1 metre.

(b) The length of the rod in the laboratory is given by

$$l = l_0 \sqrt{1 - \frac{v^2}{c^2}}$$

Here $l_0 = 1$ metre, $v = 0.8c$, therefore $l = 1 \cdot \sqrt{1 - \left(\frac{0.8c}{c}\right)^2}$

$$= 1 \cdot \sqrt{1 - 0.64} = 1 \times \sqrt{0.36} = 1 \times 0.6 = 0.6 \text{ m}$$

Ex. 4. The length of a rocket ship is 100 metres on the ground. When it is in flight its length observed on the ground is 99 metres, calculate its speed.

Solution. We have

$$l = l_0 \sqrt{1 - \frac{v^2}{c^2}} \quad \text{or} \quad \frac{l}{l_0} = \sqrt{1 - \frac{v^2}{c^2}}$$

Here

$$l_0 = 100 \text{ metres, } l = 99 \text{ metres}$$

$$\therefore \frac{99}{100} = \sqrt{1 - \frac{v^2}{c^2}}$$

or

$$v = \sqrt{\frac{(199)}{100}} c = 4.23 \times 10^7 \text{ m/s}$$

Ex. 5. A rocket of proper length 600 metres is moving directly away from earth. A light pulse sent from the earth is reflected from mirrors at the rear (back part) and front of the rocket. If the first of these reflected pulse is received back 100 seconds after emission and the second one 16 micro-second later, calculate.

(a) the distance of the rocket from earth

(b) the velocity of the rocket and

(c) its apparent length.

Solution. (a) The pulse sent from the earth reaches the rear of the rocket 50 seconds after its emission, hence at this instant the distance of the rear of the rocket from earth $= 50 \times 3 \times 10^8$ metres $= 1.5 \times 10^{10}$ metres.

(b) According to Lorentz transformation in differential form

$$\delta t' = \frac{\delta t + \frac{v \delta x}{c^2}}{\sqrt{1 - \beta^2}}$$

If the events of reception of light pulse at the rear and front of the rocket be characterised by (x_1, t_1) and (x_2, t_2) in earth frame S and (x_1', t_1') and (x_2', t_2') in the rocket frame S' moving relative to S along positive common x -axis, then given

$$\delta x' = x_2' - x_1' = 600 \text{ metres}$$

$$\delta t' = t_2' - t_1' = \frac{x_2' - x_1'}{c} = \frac{\delta x'}{c}$$

and

$$\delta t = t_2 - t_1 = \frac{16}{2} \mu \text{ sec} = 8 \mu \text{ sec} = 8 \times 10^{-6} \text{ sec.}$$

$$\delta t = \frac{\left(1 + \frac{v}{c}\right) \delta x'}{c \sqrt{1 - \frac{v^2}{c^2}}} \quad \text{or} \quad \sqrt{\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}} = \frac{c \delta t}{\delta x'}$$

$$= \frac{c \times 8 \times 10^{-6}}{600} = \frac{3 \times 10^8 \times 8 \times 10^{-6}}{600} = 4$$

$$v = 0.88c.$$

(c) The apparent length of rocket is

$$l = 600 \sqrt{1 - \frac{v^2}{c^2}} = 600 \times \sqrt{1 - (0.88)^2} = 280 \text{ m.}$$

Ex. 6. How fast would a rocket have to go relative to an observer for its length to be contracted to 99% of its length at rest. (Rohilkhand 2000, Agra 2001)

Solution. According to length contraction

$$l = l_0 \sqrt{1 - \frac{v^2}{c^2}}$$

$$\text{Here } l = \frac{99}{100} l_0 \text{ i.e., } \frac{l}{l_0} = \frac{99}{100}$$

$$\therefore \frac{99}{100} = \sqrt{1 - \frac{v^2}{c^2}} \text{ i.e., } \left(\frac{99}{100}\right)^2 = 1 - \frac{v^2}{c^2}$$

$$\frac{v^2}{c^2} = 1 - \left(\frac{99}{100}\right)^2 = \frac{(100)^2 - (99)^2}{(100)^2} = \frac{199}{(100)^2}$$

$$v = \frac{\sqrt{199}}{100} c = 0.1416c.$$

Ex. 7. A vector in system S' is represented by $8\hat{i} + 6\hat{j}$. How can the vector be represented in system S while S' is moving with velocity $0.8c\hat{i}$ with respect to S ; $\hat{i}$ and $\hat{j}$ being unit vector along the respective directions.

Solution. S' is moving relative to S with velocity $0.8c$ along x -axis; therefore the length of the vector will only change along x -axis while its length along y -axis will remain the same.

$$\text{We know } l = l_0 \sqrt{1 - \frac{v^2}{c^2}}$$

$$\text{Here } \frac{v}{c} = \frac{0.8c}{c} = 0.8 \text{ and } l_{0x} = 8$$

$$\therefore l_x = 8\sqrt{1 - 0.64} = 8\sqrt{0.36} = 8 \times 0.6 = 4.8.$$

Therefore in system S the vector may be represented by $4.8\hat{i} + 6\hat{j}$.

Ex. 8. Show that if l_0^3 is the volume of the cube, then $l_0^3 \sqrt{1 - v^2/c^2}$ is the volume viewed from a reference frame moving with uniform velocity v parallel to an edge of the cube. (Rohilkhand 1987)

Solution. Suppose two systems S and S' , S' moving with velocity v relative to S along positive direction of x -axis.

Now the volume of the cube in system $S = l_0^3$. Hence one edge of the cube $= l_0$. Let one edge of the cube be along axis of x ; thus its length along x -axis as observed by an observer in S' , $l_x = l_0 \sqrt{1 - \frac{v^2}{c^2}}$. The lengths along y and z -axes remain unchanged. Therefore volume of the cube as observed from S' .

$$l_x l_y l_z = l_0 \sqrt{1 - \frac{v^2}{c^2}} \times (l_0) \times (l_0) = l_0^3 \sqrt{1 - \frac{v^2}{c^2}}$$

Ex. 9. Calculate the percentage contraction of a rod moving with a velocity $0.8c$ in direction inclined at 60° to its own length.

Solution. Let l_0 be the length of the rod at rest.

If $0.8c$ is the velocity of the rod along the axis of x , then we have

$$l_0 = l_0 \cos 60^\circ \hat{i} + l_0 \sin 60^\circ \hat{j}$$

where $\hat{i}$ and $\hat{j}$ are unit vectors along the x and y -axes respectively.

As the contraction takes place in the direction of motion, the contraction will only take place along the axis of x ; while the component of length along the axis of y will remain the same.

If l_x is the component of the length of the rod, when in motion along the axis of x , then we have

$$l'_x = l_x \sqrt{1 - \frac{v^2}{c^2}} = (l_0 \cos 60^\circ) \sqrt{1 - \frac{v^2}{c^2}} = l_0 \times \frac{1}{2} \sqrt{1 - \left(\frac{0.8c}{c}\right)^2} = 0.31 l_0$$

The component of the length of the moving rod along y -axis is given by

$$l'_y = l_y \cos 60^\circ = \frac{\sqrt{3}}{2} l_0$$

$$\therefore \text{The length of the moving rod} = \sqrt{l'^2_x + l'^2_y} = \sqrt{\{(0.31 l_0)^2 + (\sqrt{3} l_0 / 2)^2\}} = 0.91 l_0$$

$$\therefore \text{Decrease in length of the rod due to its motion} = l_0 - 0.91 l_0 = 0.09 l_0$$

$$\text{Therefore the percentage contraction} = \frac{0.09 l_0}{l_0} \times 100 = 9\%$$

16.9. Time Dilation

Consider two systems S and S' , S' moving with velocity v relative to S along positive direction of x -axis. Let the clock be situated at x in frame S and gives signals at intervals

$$\Delta t = t_2 - t_1 \quad \dots(1)$$

The interval observed by observer in system S' will be

$$\Delta t' = t_2' - t_1' \quad \dots(2)$$

From Lorentz transformation, we have

$$t_1' = \frac{t_1 - (vx/c^2)}{\sqrt{1 - \frac{v^2}{c^2}}} \quad \dots(3)$$

and

$$t_2' = \frac{t_2 - (vx/c^2)}{\sqrt{1 - \frac{v^2}{c^2}}} \quad \dots(4)$$

Subtracting (3) from equation (4), we get

$$t_2' - t_1' = \frac{(t_2 - t_1)}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\Delta t' = \frac{\Delta t}{\sqrt{1 - \frac{v^2}{c^2}}} \quad \text{or } \Delta t' > \Delta t.$$

Thus the time interval Δt appears to the moving observer to be dilated or lengthened because the time interval in system S' is greater than that in system S . We may state **"A clock will be found to run more and more slowly if the relative velocity between the clock and the observer is increased more and more."**

The time interval measured by a clock at rest relative to observer is called the **proper interval**.

An experimental verification of time dilation : To observe time dilation in the laboratory the following conditions must be satisfied.

1. The bodies must move with relativistic speed v .
2. The events occurring in bodies should be independent of v .
3. The time interval between events should be sufficiently short so that the events may occur within a reasonably short travel of bodies, otherwise the laboratory necessary will be inconveniently long.

Some nuclear and elementary particles satisfy these three conditions, they undergo some transformation with a proper life time τ which is independent of their speed and they can be produced with large speeds so that their non-proper time τ' (measured in laboratory frame) may differ appreciably from τ .

The nuclear particles called π^+ mesons are produced with speed $0.99c$ when high energy particles generated by a synchrocyclotron strike a target. These mesons decay (break into μ mesons and neutrinos) in such a way that in every 1.8×10^{-8} sec., half of them die out. The flux of π^+ mesons was measured at two places separated by 30 metres. The laboratory time interval $\Delta t'$ for travelling this distance was given by

$$\Delta t = \frac{30\text{m}}{0.99 \times 3 \times 10^8 \text{ m/s}} = 10 \times 10^{-8} \text{ s}.$$

This is about 5.6 times of 1.8×10^{-8} s. If T is half life time of particles, then after time t , the fractional flux N/N_0 is given by

$$\frac{N}{N_0} = \left(\frac{1}{2}\right)^{t/\tau} = \left(\frac{1}{2}\right)^{5.6} = 2^{-5.6} = 2\%.$$

Hence the flux of π mesons should decrease to approximately 2% of the original flux in travelling 30 metres. But the actual flux at the second place was approximately 60% of that at first place.

This discrepancy is explained by calculating the proper time Δt by the relation.

$$\Delta t' = \frac{\Delta t}{\sqrt{1 - \frac{v^2}{c^2}}} \text{ or } \Delta t = \Delta t' \sqrt{1 - \frac{v^2}{c^2}}$$

$$= 10 \times 10^{-8} \sqrt{1 - \left(\frac{0.99c}{c}\right)^2} = 1.4 \times 10^{-8} \text{ s.}$$

This is 0.78 times of 1.8×10^{-8} sec. Hence in this much time the flux should fall to 2^{-78} 60%. This is exactly what is observed.

From this experiment it is clear that in the laboratory measurements, time taken by π^+ mesons for 30m travel is 10×10^{-8} sec; while π mesons* themselves measure the time as 1.4×10^{-8} s. Thus a 7-fold time dilation has occurred in this case.

Ex. 10. The half life of a particular particle as measured in the laboratory comes out to be 4.0×10^{-8} s when its speed is $0.8c$ and 3.0×10^{-8} s when its speed is $0.6c$. Explain this.

Solution. This can be explained on the basis of relativistic time dilation. The time interval in motion is given by

$$\Delta t' = \frac{\Delta t}{\sqrt{1 - v^2/c^2}}$$

where Δt is the proper time interval.

The proper half life of the given particle is

$$\Delta t = \Delta t' \sqrt{1 - \frac{v^2}{c^2}}$$

In first case given $\Delta t' = 4.0 \times 10^{-8}$ s. and $v = 0.8c$

$$\therefore \Delta t = 4 \times 10^{-8} \sqrt{1 - \left(\frac{.8c}{c}\right)^2} = 1.4 \times 10^{-8} \times 0.6 = 2.4 \times 10^{-8} \text{ s}$$

As proper half life is independent of velocity, therefore half life of the particle when speed is $0.6c$ must be given by

$$\Delta t' = \frac{\Delta t}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{2.4 \times 10^{-8}}{\sqrt{1 - \left(\frac{.6c}{c}\right)^2}} = \frac{2.4 \times 10^{-8}}{0.8} = 3 \times 10^{-8} \text{ s}$$

which is actual observation. Thus the variation of half life of the given particle is due to relativistic time dilation.

Ex. 11. A beam of particles of half life 2×10^{-6} second travels in the laboratory with speed 0.96 times the speed of light. How much distance does the beam travel before the flux fall to $1/2$ times the initial flux. (Kanpur 2000, Agra 1990)

Solution. The observed half life $\Delta t'$ is given by

* If τ is the mean life time, then out of N_0 initial particles, the number of particles undecayed after t is $N = N_0 e^{-t/\tau}$

$$\Delta t' = \frac{\Delta t}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{2 \times 10^{-6}}{\sqrt{1 - \left(\frac{0.96c}{c}\right)^2}} = \frac{2 \times 10^{-6}}{0.28} = 7.14 \times 10^{-6} \text{ seconds}$$

According to definition of half life in this observed time the flux falls to 1/2 times the initial value. Thus in this time the distance traversed by the beam is

$$= v \Delta t' = 0.96 \times 3 \times 10^8 \times 7.14 \times 10^{-6} = 2.1 \times 10^3 \text{ metres.}$$

Ex. 12. A particle with a mean proper life of 1 micro second (μ -s) moves through the laboratory at 2.7×10^8 m/s

- What will be its life time as measured by observer in the laboratory?
- What will be the distance traversed by it before disintegrating?
- Find the distance traversed without taking relativity into account.

Solution. (i) By the result of time dilation we have

$$\Delta t' = \frac{\Delta t}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Given $\Delta t = 1 \mu \text{ s.} = 1 \times 10^{-6} \text{ s.}$, $v = 2.7 \times 10^8 \text{ m/s.}$

$$\begin{aligned} \text{Therefore } \Delta t' &= \frac{1 \times 10^{-6}}{\sqrt{1 - \left(\frac{2.7 \times 10^8}{3 \times 10^8}\right)^2}} \\ &= \frac{3 \times 10^{-6}}{\sqrt{3^2 - (2.7)^2}} = 2.3 \times 10^{-6} \text{ s.} = 2.3 \mu \text{ s.} \end{aligned}$$

(ii) Distance traversed by the particle

$$= v \Delta t' = 2.7 \times 10^8 \times 2.3 \times 10^{-6} \text{ metres} = 620 \text{ m.}$$

(iii) Distance traversed without relativistic effects

$$= v \Delta t = 2.7 \times 10^8 \times 1 \times 10^{-6} = 270 \text{ m.}$$

Ex. 13. The proper life of π^+ is 2.5×10^{-8} seconds. If a beam of these mesons of velocity $0.8c$ is produced, calculate the distance, the beam can travel, before the flux of the meson beam is reduced to $1/e^2$ times the initial flux.

Solution. According to time-dilation, we have

$$\Delta t' = \frac{\Delta t}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Given $\Delta t = 2.5 \times 10^{-8} \text{ s.}$, $v = 0.8c.$

$$\Delta t' = \frac{2.5 \times 10^{-8}}{\sqrt{1 - \left(\frac{0.8c}{c}\right)^2}} = \frac{2.5 \times 10^{-8}}{0.6} = 4.16 \times 10^{-8} \text{ s}$$

If N_0 is the initial flux and N is the flux after time t , then we have

$$N = N_0 e^{-t/\tau}, \tau \text{ being mean life.}$$

$$\text{Here } N = \frac{1}{e^2} N_0$$

$$\therefore \frac{1}{e^2} N_0 = N_0 e^{-1/\tau}$$

$$\text{or } t = 2\tau = 2\Delta t' \text{ (Since here } \tau = \Delta t' = 4.16 \times 10^{-8} \text{ s)}$$

$\therefore$ The distance traversed by the beam before the flux of meson beam is reduce to $\frac{1}{e^2}$ times of initial flux.

$$= 2\Delta t' \times 0.8c = 8.32 \times 10^{-8} \times 0.8 \times 10^8 = 19.96 \text{ m.}$$

Ex. 14. At what speed should a clock be moved so that it may appear to lose 1 minute in each hour?

Solution. If the clock is to lose 1 minute in 1 hour, then the clock must record 59 minutes for each hour recorded by clocks stationary with respect to the observer. If v is the required speed of the clock, then according to Einstein's apparent retardation of clocks, we must have

$$59 = 60 \sqrt{(1 - v^2/c^2)}$$

$$\text{or } \frac{v}{c} = \sqrt{\left\{1 - \left(\frac{59}{60}\right)^2\right\}} = 0.18$$

$$\text{or } v = 0.18 = 0.18 \times 3 \times 10^8 = 5.4 \times 10^7 \text{ metre/second.}$$

16.10. Transformation and Addition of Velocities

Let there be two coordinate systems S and S' , the latter moving with velocity v relative to former along the positive direction of x -axis. Let a particle be moving with velocity $\vec{u}$ relative to frames S and S' respectively; then

$$\vec{u} = \hat{i}u_x + \hat{j}u_y + \hat{k}u_z$$

$$\text{and } \vec{u}' = \hat{i}'u'_x + \hat{j}'u'_y + \hat{k}'u'_z$$

where u_x, u_y and u_z are components of $\vec{u}$ along the three axes, $\hat{i}, \hat{j}$ and $\hat{k}$ being unit vectors along x, y and z -axes respectively.

u'_x, u'_y and u'_z are components of $\vec{u}'$ along respective axes. From the definition of velocity, we have

$$u_x = \frac{dx}{dt}, u_y = \frac{dy}{dt}, u_z = \frac{dz}{dt},$$

$$u'_x = \frac{dx'}{dt'}, u'_y = \frac{dy'}{dt'}, \text{ and } u'_z = \frac{dz'}{dt'}$$

Lorentz inverse transformations are given by

$$x = \frac{x' + vt'}{\sqrt{1 - \frac{v^2}{c^2}}}, y = y', z = z' \text{ and } t = \frac{t' + (vx'/c^2)}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Differentiating these transformation equations, we get

$$dx = \frac{dx' + vdt'}{\sqrt{1 - \frac{v^2}{c^2}}}, dy = dy', dz = dz' \text{ and } dt = \frac{dt' + (vdx'/c^2)}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Now from these expressions, we have

$$u_x = \frac{dx}{dt} = \frac{dx' + v dt'}{dt' + \frac{v}{c^2} dx'} = \frac{\frac{dx'}{dt'} + v}{1 + \frac{v}{c^2} \frac{dx'}{dt'}}$$

$$u_x = \frac{u_x' + v}{1 + \frac{v}{c^2} u_x'} \quad \dots(5)$$

$$u_y = \frac{dy}{dt} = \frac{dy' \sqrt{\left(1 - \frac{v^2}{c^2}\right)}}{dt' + \frac{v}{c^2} \frac{dx'}{dt'}} = \frac{\frac{dy'}{dt'} \sqrt{\left(1 - \frac{v^2}{c^2}\right)}}{1 + \frac{v}{c^2} \frac{dx'}{dt'}}$$

$$u_y = \frac{u_y' \sqrt{\left(1 - \frac{v^2}{c^2}\right)}}{1 + \frac{v}{c^2} u_x'} \quad \dots(6)$$

$$\text{Similarly } u_z = \frac{u_z' \sqrt{\left(1 - \frac{v^2}{c^2}\right)}}{1 + \frac{v}{c^2} u_x'} \quad \dots(7)$$

Equations (5), (6) and (7) represent transformation of velocity components from one inertial system to another.

If we consider the particle to be a photon moving with velocity c in frame S' and the system S' to be moving with velocity v relative to system S along positive direction of x -axis, then from equation (5), we have $u_x = \frac{c + v}{1 + v \cdot c/c^2} = c$, i.e., *the speed of light is same for all inertial frames whatever their relative speed may be and no particle can attain a speed greater than the speed of light.*

Criticism of Velocity—Addition Theorem. Though velocity addition theorem of special relativity imposes an upper limit on velocities of particles, this upper limit is the velocity of light c , there exist examples in which velocities greater than c exist:

1. The moving material particles whose refractive index μ is less than 1 provide one such example, the wave velocity μ of such material particles is greater than the velocity of light c . But in such cases it is the group velocity with which signals can be propagated and it has been shown by Sommerfeld that group velocity is less than or equal to c for all media.

2. Consider a long scale making a very small angle θ with x -axis of a cartesian coordinate system. Now let the scale move along y -axis with velocity v , then the point of intersection of the scale with the x -axis would move with a velocity u such that

$$\tan \theta = \frac{v}{u} \text{ or } u = \frac{v}{\tan \theta}$$

Now as θ be made infinitely smaller and smaller, u would achieve greater and greater values and that it may exceed c , depending on the value of θ .

Such examples of velocities are only of geometrical significance and can play no role in physical world.

3. Recently Prof. E.C.G. Sudershan has found the existence of particles faster than light.

These particles have imaginary mass and are known as "*tachyons*"*. The possibility of existence of tachyons is not denied by special theory of relativity. The existence of these particles has created a great interest in their studies in recent years.

However we may still state : "*No material particle can have the speed greater than the speed of light.*"

16.11. Fresnel's Coefficient of Drag

If light travels in a moving medium, the light rays are dragged by the medium. This phenomenon was verified by Fizeau and may be explained by velocity-addition theorem.

The Fresnel coefficient is defined as the ratio of change in velocity of light in medium to the velocity of medium, i.e.

$$\mu_d = \frac{\text{Change in velocity of light in medium}}{\text{Velocity of medium}}$$

Consider two systems S and S' , S' moving with velocity v relative to S along positive direction of X -axis. We may assume S' to be a moving medium (like water) and the light signal in water, then velocity of light in stationary water $u_x' = \frac{c}{n}$ where n is refractive index of medium.

Velocity of light as observed from S ,

$$\begin{aligned} u_x &= \frac{u_x' + v}{1 + \frac{u_x' v}{c^2}} = \frac{\frac{c}{n} + v}{1 + \left(\frac{c}{n}\right) \frac{v}{c^2}} \\ &= \left(\frac{c}{n} + v\right) \left(1 + \frac{v}{cn}\right)^{-1} \quad (\text{As } v \ll c) \\ &= \left(\frac{c}{n} + v\right) \left(1 - \frac{v}{cn}\right) \\ &= \frac{c}{n} + v - \frac{v^2}{n^2} - \frac{v^2}{cn} \\ &= \frac{c}{n} + v - \frac{v^2}{n^2} \quad (\text{since } v \ll c) \\ &= \frac{c}{n} + v \left(1 - \frac{1}{n^2}\right) \\ &= u_x' + v \left(1 - \frac{1}{n^2}\right) \end{aligned}$$

$$\text{Increase in velocity } u_x - u_x' = v \left(1 - \frac{1}{n^2}\right)$$

$$\therefore \text{Coefficient of drag, } \mu_d = \frac{u_x - u_x'}{v} = \left(1 - \frac{1}{n^2}\right)$$

The expression was theoretically obtained by Fresnel; so it is called Fresnel's coefficient of drag.

Ex. 15. An electron is moving with a speed of $0.85c$ in a direction opposite to that of moving photon. Calculate the relative velocity of the electron and photon.

Solution. The speed of photon = c

The speed of electron = $0.85c$.

* Bilanuk, Deshpande and Sudershan Am, Phy, 30, 718 (1962) : M. Arone and E.C.G. Sudershan Phy Rev. 173-1622 (1968); Dhar and Sudershan Phy. Rev. 174, 1808 (1968)

Let the photon and electron be moving along (+)ve and (-) ve directions of x -axis respectively. Let the electron moving with velocity $-0.85c$ be at rest in system S . Then we may assume that system S' or laboratory is moving with velocity $0.85c$ relative to system S (electron). Then we may write $v = 0.85c$, $u'_x = c$.

$$\therefore u_x = \frac{u'_x + v}{1 + \frac{u'_x v}{c^2}} = \frac{c + 0.85c}{1 + \frac{0.85c(c)}{c^2}} = \frac{c + 0.85c}{1 + 0.85} = c.$$

Therefore relative velocity of electron and photon = c .

Ex. 16. Two particles come towards each other with speed $0.8c$ with respect to laboratory. What is their relative speed?

Solution. The problem consists of two velocities -0.8 and $0.8c$ observed from laboratory, therefore to find the required velocity we have to use the theorem of composition of velocities. For the solution of the problem consider a system S in which the particle having velocity $-0.8c$ is at rest. Thus we may say that system S' , i.e., laboratory is moving with velocity $+0.8c$ relative to system S , i.e., $v = 0.8c$. Now we have to find the velocity of particle moving with velocity $+0.8c$ relative to system S' (or laboratory) in system S , we have from the theorem of composition of velocities.

$$u_x = \frac{u' + v}{1 + u'v/c^2}$$

Here $u' = 0.8c$, $v = 0.8c$.

$$\text{Therefore } u_x = \frac{0.8c + 0.8c}{1 + \frac{(0.8c)(0.8c)}{c^2}} = \frac{0.8c + 0.8c}{(1 + 0.64)} = \frac{1.6c}{1.64} = 0.9755c.$$

Thus the relative speed of particles = $0.9755c$.

Ex. 17. Two β -particles A and B travel in opposite directions each with a velocity $0.9c$. What is their relative velocity (i) as observed by A and (ii) as observed by a stationary observer. (CSE 2008, Kanpur 1984)

Solution. This problem is analogous to problem 16.

$$\therefore u_x = \frac{u' + v}{1 + \frac{u'v}{c^2}} = \frac{0.9c + 0.9c}{1 + \frac{(0.9c)(0.9c)}{c^2}} = \frac{1.8c}{1 + .81} = \frac{1.8}{1.81} c = 0.99c.$$

$\therefore$ Relative speed of particles in both cases (i) and (ii) is $0.99c$.

Ex. 18. A particle moves with velocity represented by a vector $\mathbf{u}' = 3\hat{i} + 4\hat{j} + 12\hat{k}$ metre/sec in frame S' . Find the velocity of the particle in frame S if S' moves with velocity $0.8c$ relative to S along (+) x -axis.

Solution. If S and S' are two systems, the latter moving with velocity v relative to S along (+)ve direction of x -axis, then we have from the theorem of composition of velocities

$$u_x = \frac{u'_x + v}{1 + \frac{u'_x v}{c^2}}, \quad u_y = \frac{u'_y \sqrt{1 - \frac{v^2}{c^2}}}{\left(1 + \frac{v}{c^2} u'_x\right)}, \quad u_z = \frac{u'_z \sqrt{1 - \frac{v^2}{c^2}}}{1 + \frac{v}{c^2} u'_x}$$

$$\text{Here } \mathbf{u}' = 3\hat{i} + 4\hat{j} + 12\hat{k} = \hat{i}u'_x + \hat{j}u'_y + \hat{k}u'_z$$

$$\therefore u'_x = 3, u'_y = 4, u'_z = 12 \text{ and } \mathbf{u} = \hat{i}u_x + \hat{j}u_y + \hat{k}u_z, \text{ also } v = 0.8c$$

Thus we have

$$u_x = \frac{u'_x + v}{1 + \frac{v}{c^2} u'_x} = \frac{3 + 0.8c}{1 + \frac{0.8c}{c^2} \times 3} = \frac{3 + 0.8 \times 3 \times 10^8}{1 + \frac{0.8 \times 3}{3 \times 10^8}} \quad (\text{since } c = 3 \times 10^8 \text{ m/s.})$$

$$= \frac{3 + 2.4 \times 10^8}{10^8 + 0.8} \times 10^8 = 2.4 \times 10^8$$

$$u_y = \frac{u'_y \sqrt{1 - v^2/c^2}}{1 + \frac{v}{c^2} u'_x} = \frac{4\sqrt{1 - (0.8c/c)^2}}{1 + \frac{0.8c}{c^2} \times 3} = \frac{4\sqrt{(0.2 \times 1.8)}}{1 + \frac{0.8 \times 3}{3 \times 10^8}} = 2.4$$

$$\text{Similarly } u_z = \frac{u'_z \sqrt{1 - v^2/c^2}}{1 + \frac{v}{c^2} u'_x} = \frac{12\sqrt{1 - (0.8c/c)^2}}{1 + \frac{0.8c}{c^2} \times 3} = 7.2$$

Thus we have velocity $\mathbf{u}$ of the particle in the frame S to be

$$= 2.4 \times 10^8 \hat{i} + 2.4 \hat{j} + 7.2 \hat{k}.$$

Ex. 19. A scientist observes that a certain atom A moving relative to him with velocity 2×10^8 m/s. emits a particle B which moves with velocity 2.8×10^8 m/s. with respect to atom. Calculate the velocity of the emitted particle relative to scientist.

Solution. According to velocity addition theorem, we have

$$u_x = \frac{u' + v}{1 + u'v/c^2}$$

Let us consider the particle A to be in system S' and scientist in system S . Then $u' =$ velocity of the particle B relative to $S' = 2.8 \times 10^8$ m/s.

$v =$ velocity of system S' relative to $S = 2 \times 10^8$ m/s

$u' =$ velocity of the particle B relative $S = ?$

$$\therefore u' = \frac{2.8 \times 10^8 + 2 \times 10^8}{1 + \frac{(2.8 \times 10^8) \times (2 \times 10^8)}{(3 \times 10^8)^2}} = \frac{4.8 \times 10^8}{1 + \frac{5.6}{9}} = \frac{4.1 \times 10^8}{1.622} = 2.95 \times 10^8 \text{ m/s.}$$

16.12. Relativistic Mechanics; Lorentz Conservation of Momentum.

We have seen that the formulation of Lorentz transformation is based on the requirement that the laws of electrodynamics (in particular Maxwell's equations) remain invariant. The electromagnetic disturbance is propagated with speed c ($= 3 \times 10^8$ m/s.) in free space, independent of the choice of reference system and *no signal can travel with a speed greater than this speed c .*

According to Newton's third law "action and reaction are equal and opposite and take place at the same time and not action today with reaction tomorrow". Let us consider two bodies, say stars A and B , which are far apart. The force between them is the mutual gravitational attraction. Now if A moves slightly, the force on B changes and it moves accordingly. This is the action of A on B . The movement of B in turn changes the force on A , causing it to move, which is the reaction of B on A . According to Newton's third law action and reaction should

take place simultaneously which means that the signal (which is gravitational force in this case) should travel with infinite speed. However, infinite signals are denied by the special theory of relativity. Therefore, Newton's third law contradicts the special theory of relativity. It, therefore, becomes necessary to find equations of mechanics which remain invariant under Lorentz transformations. Here we shall try to retain the principle of conservation of linear momentum under Lorentz transformations; since this principle is regarded to have universal validity independent of its derivation from Newtonian mechanics. According to the principle of conservation of linear momentum, "*The linear momentum of a system remains conserved if no external forces act on the system.*". Accordingly if two bodies collide in the absence of external forces, the change in momentum of one body is equal and opposite to that of the other. The definition of momentum given in classical mechanics in which we assume that the mass of a moving body to be the same as that of stationary one, does not hold if we examine this from Lorentz transformations, but due to universal validity of conservation of momentum, it must be conserved under Lorentz transformation equations if we consider them to be valid. Thus for the conservation of momentum in Lorentz transformations, we have to deal with the problem in such a way that the law of conservation of momentum is retained. For the purpose we consider the hypothetical experiment first suggested by Tolman and Leks, by which we conclude, "*For the Lorentz conservation of momentum, we have to rule out our classical concept that mass of a moving body is equal to that of a stationary body and we have to accept that the mass of body changes with velocity.*"

16.13. Variation of Mass with Velocity

Let us first consider the hypothetical experiment of Tolman and Leks. Consider two systems of co-ordinates S and S' , the latter moving with velocity v relative to S along (+)ve direction of their common x -axis. We assume that S' is approaching S with velocity v as shown in Fig. 16-10.

Let there be two perfectly elastic particles P and Q such that P in system S is moving with velocity u_y along (+) y -axis and Q in system S' is moving with velocity $u_{y'}$ along (+)ve y -axis i.e., the particles have equal and opposite velocities. Let velocity u_y be non-relativistic while velocity v is relativistic. The masses of the two particles are equal when measured from the same system. Let m_0 be the mass of each particle at rest. Let the particles collide due to velocity of system S' in such a way that the line of their centres is along y -axes, i.e., the collision has no effect on the components of velocity along x or z -axis in its own system. Since collision is elastic the direction

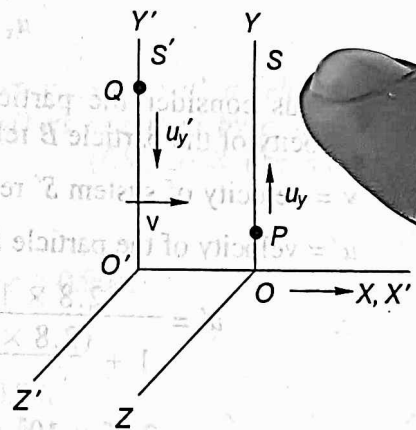


Fig. 16-10

of motion of particles is reversed as seen from its own system. after impact velocity of P is $-u_y$ and that of Q is $u_{y'}$ because in elastic collision of equal masses; the velocities are interchanged. Now the velocity components *before collision* as measured by observer in S according to law of transformation of velocities.

$$\text{i.e., } \left[u_x = \frac{u_{x'} + v}{1 + u_{x'}v/c^2}, u_y = \frac{u_{y'}\sqrt{1 - v^2/c^2}}{1 + u_{x'}v/c^2} \text{ and } u_z = \frac{u_{z'}\sqrt{1 - v^2/c^2}}{1 + u_{x'}v/c^2} \right]$$

we have for particle P , $u_{xP} = 0$, $u_{yP} = u_y$ and $u_{zP} = 0$

For particle Q , $u_{xQ} = v$, $u_{yQ} = -u_{y'}\sqrt{1 - (v^2/c^2)}$, $u_{zQ} = 0$

[since $u_{x'} = 0$, $u_{y'} = -u_y$ and $u_{z'} = 0$]

Thus the resultant momentum of whole system before impact along y-axis as viewed from system S.

$$= m_0 u_y - mu_y \sqrt{1 - v^2/c^2}$$

where m is the mass of particle moving with velocity v .

Now **after impact** when the velocities of two particles P and Q are given by

For Particle P : $u_{xP} = 0$, $u_{yP} = -u$ and $u_{zP} = 0$

For Particle Q : $u_{xQ} = v$, $u_{yQ} = u_y \sqrt{1 - (v^2/c^2)}$, $u_{zQ} = 0$

Thus the resultant momentum after impact as viewed from system S along y-axis.

$$= -m_0 u_y + mu_y \sqrt{1 - v^2/c^2}$$

But according to principle of conservation of momentum

Momentum before impact = Momentum after impact.

$$\text{i.e.,} \quad m_0 u_y - mu_y \sqrt{1 - v^2/c^2} = -m_0 u_y + mu_y \sqrt{1 - v^2/c^2} \quad \dots(1)$$

$$\text{We, thus have } (m_0 u_y + m_0 u_y) = (m u_y + m u_y) \sqrt{1 - \frac{v^2}{c^2}}$$

$$\text{or} \quad 2m_0 u_y = 2m u_y \sqrt{1 - \frac{v^2}{c^2}}$$

$$\text{Therefore} \quad m_0 = m \sqrt{1 - v^2/c^2}$$

$$\text{or} \quad m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

Thus the law of conservation of momentum leads to a *very important conclusion*. "**mass of a body increases with increase of velocity**." Now if we express the momentum as $mv =$

$\frac{m_0 v}{\sqrt{1 - v^2/c^2}}$, it will be conserved under Lorentz transformations. In this problem we have

neglected the change in mass due to velocity u_y —actually it is justified since we have assumed u_y to be non-relativistic, (i.e., very-very small in comparison to the velocity of light).

Ex. 20. Find the velocity at which the mass of a particle is double its rest mass.

(Kakatiya Univ., 1988)

Solution. The mass of a particle (rest mass m_0) moving with velocity v is given by

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

Given $m = 2m_0$

$$2m_0 = \frac{m_0}{\sqrt{1 - v^2/c^2}} \quad \text{or} \quad 2 = \frac{1}{\sqrt{1 - v^2/c^2}}$$

or

$$1 - \frac{v^2}{c^2} = \frac{1}{4} \quad \text{or} \quad \frac{v^2}{c^2} = \frac{3}{4}$$

$\therefore$

$$v = \frac{\sqrt{3}}{2} c = \frac{1.732}{2} \times 3 \times 10^8 \text{ m/s} = 2.598 \times 10^8 \text{ m/s.}$$

Ex. 21. The rest mass of an electron is 9.1×10^{-31} kg. What will be its mass if it were moving with 4/5th the speed of light ?

(Bombay 2005, Kanpur 1987)

Solution. The mass of the electron if it were moving with speed v is given by

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}, \text{ where } m_0 \text{ is the rest mass of the electron.}$$

Here $v = \frac{4}{5}c = 0.8c$ and $m_0 = 9.1 \times 10^{-31}$ kg.

$$\begin{aligned} \therefore m &= \frac{9.1 \times 10^{-31}}{\sqrt{1 - \left(\frac{0.8c}{c}\right)^2}} = \frac{9.1 \times 10^{-31}}{\sqrt{1 - 0.64}} = \frac{9.1 \times 10^{-31}}{\sqrt{0.36}} \\ &= \frac{9.1 \times 10^{-31}}{(0.6)} = 15.16 \times 10^{-31} \text{ kg.} \end{aligned}$$

16-14. Relativistic form of Newton's Second Law

Newton's second law states "The rate of change of linear momentum ($\vec{p}$) of a particle is equal to force ($\vec{F}$) impressed on it", i.e.

$$\vec{F} = \frac{d\vec{p}}{dt} \quad \dots(1)$$

If m is the mass of the particle moving with velocity $\vec{v}$ and m_0 its rest mass, then

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{m_0}{\sqrt{1 - \frac{\vec{v} \cdot \vec{v}}{c^2}}}$$

$$\text{Momentum } \vec{p} = m\vec{v} = \frac{m_0\vec{v}}{\sqrt{1 - \frac{\vec{v} \cdot \vec{v}}{c^2}}} \quad \dots(2)$$

Substituting the value in equation (1), we get the

$$\begin{aligned} \vec{F} &= \frac{d}{dt} \left[\frac{m_0\vec{v}}{\sqrt{1 - \frac{\vec{v} \cdot \vec{v}}{c^2}}} \right] = \frac{d}{dt} \left[m_0\vec{v} \left(1 - \frac{\vec{v} \cdot \vec{v}}{c^2} \right)^{-1/2} \right] \\ &= m_0 \left[\frac{d\vec{v}}{dt} \left(1 - \frac{\vec{v} \cdot \vec{v}}{c^2} \right)^{-1/2} + \frac{\vec{v}}{2c^2} \left(1 - \frac{\vec{v} \cdot \vec{v}}{c^2} \right)^{-3/2} \times \left(\vec{v} \cdot \frac{d\vec{v}}{dt} + \vec{v} \cdot \frac{d\vec{v}}{dt} \right) \right] \\ &= m_0 \left(1 - \frac{\vec{v} \cdot \vec{v}}{c^2} \right)^{-3/2} \left[\left(1 - \frac{\vec{v} \cdot \vec{v}}{c^2} \right) \frac{d\vec{v}}{dt} + \frac{\vec{v}}{c^2} \left(\vec{v} \cdot \frac{d\vec{v}}{dt} \right) \right] \end{aligned}$$

If force is parallel to velocity, then $\vec{v}$ and $\frac{d\vec{v}}{dt}$ are along the same direction, so

$$\begin{aligned} \vec{F} &= m_0 \left(1 - \frac{v^2}{c^2} \right)^{-3/2} \left[\frac{dv}{dt} \left(1 - \frac{v^2}{c^2} + \frac{v^2}{c^2} \right) \right] \\ &= m_0 \frac{dv}{dt} \left(1 - \frac{v^2}{c^2} \right)^{-3/2} \quad \dots(3) \end{aligned}$$

This is required expression.

16-15. Equivalence of Mass and Energy

From work-energy theorem, the kinetic energy of a moving body is equal to the work done by the external force that imparts the velocity to the body from rest. If F is the force acting on

the body, then work done by the force on body in raising its velocity from $v = 0$ to $v = v$ is given by

$$W = \int_0^v \mathbf{F} \cdot d\mathbf{s}.$$

Kinetic energy of the body $E_k = T$ (say) $= W$

$$= \int_0^v \mathbf{F} \cdot \frac{d\mathbf{s}}{dt} dt = \int_0^v \mathbf{F} \cdot \mathbf{v} dt$$

$$= \int_0^v \frac{d\mathbf{p}}{dt} \cdot \mathbf{v} dt \quad \left(\text{Since } \mathbf{F} = \text{rate of change of momentum} = \frac{d\mathbf{p}}{dt} \right)$$

$$= \int_0^v \mathbf{v} \cdot \frac{d}{dt} (m \mathbf{v}) dt \quad \left(\text{Since } \mathbf{p} = m \mathbf{v}; m \text{ being the mass of moving body} \right)$$

$$= \int_0^v \mathbf{v} \cdot d(m\mathbf{v}) \quad \dots (1)$$

But from the relation of variation of mass with velocity, the mass of body in motion,

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}, m_0 \text{ being rest mass of body}$$

$$\therefore E_k = T = \int_0^v \mathbf{v} \cdot d \left(\frac{m_0}{\sqrt{1 - v^2/c^2}} \cdot \mathbf{v} \right) = m_0 \int_0^v \mathbf{v} \cdot d [v (1 - v^2/c^2)^{-1/2}]$$

$$= m_0 \int_0^v \mathbf{v} \cdot \left[(1 - v^2/c^2)^{-1/2} dv - v \cdot \frac{1}{2} (1 - v^2/c^2)^{-3/2} \times \left(-\frac{2v}{c^2} dv \right) \right]$$

$$= m_0 \int_0^v v (1 - v^2/c^2)^{-3/2} \left[1 - \frac{v^2}{c^2} + \frac{v^2}{c^2} \right] dv = m_0 \int_0^v \frac{v dv}{(1 - v^2/c^2)^{3/2}}$$

Substituting $1 - \frac{v^2}{c^2} = y$ (i.e., $-\frac{2v}{c^2} dv = dy$), we get

$$E_k = m_0 \int_0^{(1 - v^2/c^2)} \frac{(-c^2/2) dy}{y^{3/2}} = \frac{m_0 c^2}{2} \left[\frac{y^{-1/2}}{-1/2} \right]_0^{1 - v^2/c^2}$$

$$= m_0 c^2 \left[\frac{1}{\sqrt{1 - v^2/c^2}} - 1 \right] = \left[\frac{m_0}{\sqrt{1 - v^2/c^2}} - m_0 \right] c^2 = (m - m_0) c^2 \quad \dots (2)$$

This is a relativistic expression for kinetic energy. Thus the kinetic energy for a moving body is equal to gain in mass times the square of speed of light. Therefore $m_0 c^2$ may be regarded as rest energy of body of rest mass m_0 . This rest energy may be considered as

internal stored energy to the body. Then total energy (E) of the body is the sum of rest energy and kinetic energy of the body i.e.,

$$E = \text{rest energy} + \text{relativistic kinetic energy} = m_0 c^2 + (m - m_0) c^2 = mc^2$$

$$\therefore E = mc^2$$

This is well known Einstein mass energy relation which states a universal equivalence between mass and energy.

Limiting value of Relativistic Kinetic energy :

Relativistic kinetic energy is $E_k = T = (m - m_0) c^2$

$$\left(\frac{m_0}{\sqrt{1 - v^2/c^2}} - m_0 \right) c^2 = [(1 - v^2/c^2)^{-1/2} - 1] m_0 c^2$$

$$= \left[1 + \frac{1}{2} \frac{v^2}{c^2} + \frac{3}{8} \frac{v^4}{c^4} + \dots - 1 \right] m_0 c^2$$

If $v \ll c$ i.e. $v/c \ll 1$ (classical limit); then equation (4) takes the form

$$E_k = T = \frac{1}{2} \frac{v^2}{c^2} \times m_0 c^2 = \frac{1}{2} m_0 v^2 \quad \dots(4)$$

which is well known expression of kinetic energy in non-relativistic physics.

Thus we conclude that for relativistic cases we may use the expression $\frac{1}{2} mv^2$ while for relativistic cases we must use expression $(m - m_0) c^2$ for kinetic energy.

It is to be noted as $v \rightarrow c$, i.e., $v/c \rightarrow 1$; the kinetic energy T tends to infinity. That means an infinite amount of work would be needed to be done on the particle to accelerate it up to the speed of light. Once again we find that c plays the role of a limiting velocity.

16-16. Evidences in Support of Mass Energy Relation

The mass energy relation has been experimentally verified for all types of energy. For example.

(1) We have kinetic energy $T = mc^2 - m_0 c^2$

$$\text{As } m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\therefore T = \frac{m_0 c^2}{\sqrt{1 - \frac{v^2}{c^2}}} - m_0 c^2 = m_0 c^2 \left[\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} - 1 \right]$$

The relation consists merely $m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$. As this formula is verified, mass

energy relation of eqn. (1) is verified indirectly.

(2) Fine structure of H_α line could only be explained by Sommerfeld on the basis of elliptical orbits of electrons around the nucleus by introducing relativistic correction for energy due to change in velocity of these electrons.

(3) In the process of annihilation of matter a system consisting of two particles, an electron and a positron, may give up all of its mass producing two photons or radiant energy. Thus the whole mass is converted into energy due to universal equivalence of mass and energy. This verifies the relation for *radiant energy*.

(4) During fusion process of two neutrons and protons to form ${}^4_2\text{He}$ nucleus, enormous energy is released, which may be seen as follows :

The Mass of two protons

$$= 2 \times \text{mass of hydrogen nucleus} = 2m({}_1\text{H}^1)$$

$$= 2 \times 1.00815 \text{ a.m.u.}$$

$$= 2.01630 \text{ a.m.u.}$$

Mass of two neutrons = $2 \times \text{mass of neutron} = 2m({}_0\text{n}^1)$

$$= 2 \times (1.00898) \text{ a.m.u.}$$

$$= 2.01796 \text{ a.m.u.}$$

$$\therefore 2m({}_1\text{H}^1) + 2m({}_0\text{n}^1) = (2.01630 + 2.01796) \text{ a.m.u.} = 4.03426 \text{ a.m.u.}$$

$\therefore$ The difference in mass of the ${}^4_2\text{He}$ nucleus and the total mass of the constituent nucleon in free state is

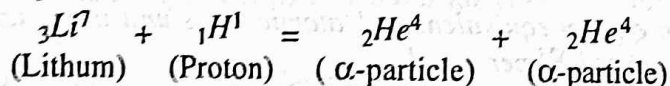
$$(4.03426 - 4.00387) \text{ a.m.u.} = 0.03039 \text{ a.m.u.}$$

$$= 0.03039 \times 931.1 \text{ MeV. (since } 1 \text{ a.m.u.} = 931.1 \text{ MeV)}$$

$$= 28.3 \text{ MeV.}$$

This explains why during fusion process of two neutrons and two protons, enormous amount of energy (= 28.3 MeV) is released. This is one of the basis of the explanations for the source of solar energy.

Another nuclear reaction studied by J.D. Cockcroft and G.T.S. Walton is that in which ${}^7_3\text{Li}$ nucleus was bombarded by protons of energy ranging between (0.5 – 1) MeV, with the production of 2α -particles, i.e.



The masses involved in this reaction are known to a high degree of accuracy and are given below :—

$$m({}_3\text{Li}^7) = 7.01823 \text{ a.m.u.}$$

$$m({}_1\text{H}^1) = 1.00815 \text{ a.m.u.}$$

and

$$m({}_2\text{He}^4) = 4.00387 \text{ a.m.u.}$$

$$\therefore m({}_3\text{Li}^7) + m({}_1\text{H}^1) = 8.02638 \text{ a.m.u.}$$

and

$$m({}_2\text{He}^4) + m({}_2\text{He}^4) = 8.00774 \text{ a.m.u.}$$

$$\therefore \text{Decrease in mass} = (8.02638 - 8.00774) \text{ a.m.u.} = 0.01864 \text{ a.m.u.}$$

which is equivalent to an energy $(0.01864 \times 931.1) \text{ MeV}$

$$= 17.35 \text{ MeV}$$

This is a fair agreement with experimental value of 17.28 MeV.

(5) In nuclear fission the heavy nuclei of certain substances like uranium may form fission fragments. The total rest mass of these fragments is less than the mass of original heavy nucleus. The difference in rest mass Δm appears in the form of equivalent energy $\Delta E = \Delta m \cdot c^2$ according to the mass energy relation. The liberation of enormous energy due to the decrease in mass is the basic principle of atom bomb.

(6) When high energy radiations are absorbed by matter under certain conditions, material particles are produced. In the process of pair production the energy is converted into mass, for example, electron positron pairs in cosmic ray showers; thus verifying the equivalence of mass and energy.

(7) Due to radiation of energy, there may be loss in the mass of the stars.

(8) Nuclei of atoms possess protons and neutrons bound together by strong attractive forces. Thus energy must be supplied in order to break apart a nucleus into its constituent nucleons. This energy, called the *binding energy of the nucleus*, has an equivalent mass $\Delta m = \frac{\Delta E}{c^2}$. Therefore on the basis of mass energy relation we expect a nucleus to have a mass smaller

than the mass of its constituent nucleons by an amount $\frac{\Delta E}{c^2}$. This view is confirmed by actual mass spectrographic measurements, the atomic masses always fall short of the masses of the constituent particles.

The *mass defect* Δm of a nucleus of atomic number Z and atomic weight A is given by

$$\Delta m = Z \cdot m_H + (A - Z) \cdot m_n - M_{Z,A}$$

where m_H is the mass of a hydrogen atom, m_n is the mass of a neutron and $M_{Z,A}$ is the observed mass of the atom under consideration.

Consider the example of deuteron ${}_1H^2$ where $Z = 1, A = 2$ and

$$M_{1,2} = \text{mass of deuteron} = 3.3442 \times 10^{-27} \text{ kg.}$$

$$\text{Also } m_H = 1.67338 \times 10^{-27} \text{ kg., } m_n = 1.6748 \times 10^{-27} \text{ kg.}$$

So that the *mass defect of deuteron*,

$$(\Delta m)_{H_2} = 1 m_H + 1 m_n - M_{H_2} = 3.98 \times 10^{-30} \text{ kg.}$$

the equivalent energy,

$$\begin{aligned} \Delta E &= \Delta m \cdot c^2 = 3.582 \times 10^{-13} \text{ joules} \\ &= 2.238 \text{ MeV.} \end{aligned}$$

which reasonably agrees with the experimentally measured value 2.226 MeV of the binding energy of deuteron nucleus. *This verifies mass-energy relation for binding energy of the nucleus.*

Ex. 22. Calculate the energy equivalent to 1 atomic mass unit in million electron-volt. Given Avogadro Number = 6×10^{23} per g mol

Solution. We know $E = mc^2$.

$$\text{Here } m = 1 \text{ a.m.u.} = \frac{1}{6 \times 10^{23}} \text{ g} = \frac{1}{6 \times 10^{26}} \text{ kg and } c = 3 \times 10^8 \text{ m/s.}$$

$$\begin{aligned} \therefore E &= \left(\frac{1}{6 \times 10^{26}} \right) \times (3 \times 10^8 \text{ m/s.})^2 = \frac{9 \times 10^{16}}{6 \times 10^{26}} \text{ joules} \\ &= \frac{9 \times 10^{16}}{6 \times 10^{26} \times 1.6 \times 10^{-19}} \text{ eV} = 937 \times 10^6 \text{ eV} = 937 \text{ MeV.} \end{aligned}$$

Ex. 23. Calculate the rest energy of electron and proton in MeV.

(Given $m_e = 9.11 \times 10^{-31} \text{ kg}$ and $m_p = 1.673 \times 10^{-27} \text{ kg}$.) (Osmania 1991)

Solution. Einstein mass-energy equivalence relation is $E = mc^2$.

For electron, $m_e = 9.11 \times 10^{-31} \text{ kg}$.

$$E_e = m_e c^2 = 9.11 \times 10^{-31} \times (3 \times 10^8)^2 \text{ joules}$$

$$= \frac{9.11 \times 10^{-31} \times (3 \times 10^8)^2}{1.6 \times 10^{-13}} \text{ MeV} = 0.512 \text{ MeV}$$

For proton, $m_p = 1.673 \times 10^{-27} \text{ kg}$.

$$E_p = m_p c^2 = 1.673 \times 10^{-27} \times (3 \times 10^8)^2 \text{ joules}$$

$$= \frac{1.673 \times 10^{-27} \times (3 \times 10^8)^2}{1.6 \times 10^{-13}} \text{ MeV} = 914 \text{ MeV.}$$

Ex. 24. Calculate the fractional change in the mass of the hydrogen atom when it is ionized from the following data :

The binding energy of hydrogen atom = 13.58 eV.

The rest mass of hydrogen atom, $m_0 = 1.00797 \text{ a.m.u.}$

Solution. The change in rest mass $\Delta m_0 = \frac{\Delta E}{c^2}$

$$= \frac{13.58 \text{ eV}}{(3 \times 10^8 \text{ m/s})^2} = \frac{13.58 \times 1.6 \times 10^{-19} \text{ joules}}{(3 \times 10^8 \text{ m/s})^2}$$

$$= \frac{13.58 \times 1.6 \times 10^{-19}}{(3 \times 10^8 \text{ m/s})^2} \text{ kg.} = \frac{13.58 \times 1.6 \times 10^{-19}}{(3 \times 10^8)^2 \times 1.66 \times 10^{-27}} \text{ amu.}$$

$$= 1.46 \times 10^{-8} \text{ a.m.u.}$$

So fractional change in the mass of the hydrogen atom,

$$\frac{\Delta m_0}{m_0} = \frac{1.46 \times 10^{-8}}{1.00797} \text{ kg.} = 1.45 \times 10^{-8} = 1.45 \times 10^{-6} \%$$

Ex. 25. How much mass is lost when 1 kg of water at 0°C turns to ice at 0°C ?

(Bombay 1998, Agra 1995)

Solution. The heat energy lost by water

$$= (1 \text{ kg}) \times (80 \text{ kilocal/kg.}) = 80 \text{ kilocal} = 80 \times 4.2 \times 10^3 \text{ joules}$$

$\therefore$ Equivalent loss in mass,

$$\Delta m = \frac{\Delta E}{c^2} = \frac{80 \times 4.2 \times 10^3}{(3 \times 10^8)^2} \text{ kg} = 3.73 \times 10^{-12} \text{ kg.}$$

Ex. 26. Find the velocity that an electron must be given so that its momentum is 10 times its rest mass times the speed of light. What is the energy at this speed.

(Rohilkhand 1999, Agra 1992)

Solution. The momentum of an electron of rest mass m_0 moving with velocity v is given by

$$p = \frac{m_0 v}{\sqrt{1 - v^2/c^2}}$$

$$\text{Given } \frac{m_0 v}{\sqrt{1 - \frac{v^2}{c^2}}} = 10m_0 c \text{ or } \frac{v}{c} = 10 \sqrt{1 - \frac{v^2}{c^2}}$$

$$\text{Squaring we get } \frac{v^2}{c^2} = 100 \left(1 - \frac{v^2}{c^2}\right) = 100 - 100 \frac{v^2}{c^2}$$

$$\text{or } (100 + 1) \frac{v^2}{c^2} = 100$$

$$\text{or } \frac{v^2}{c^2} = \frac{100}{111} \quad \dots (1)$$

$$\text{or } \frac{v}{c} = \sqrt{\left(\frac{100}{111}\right)} = 0.995$$

$$\therefore v = 0.995c = 0.995 \times 3 \times 10^8 \text{ m/s.} = 2.985 \times 10^8 \text{ m/s.}$$

The mass of the electron at this speed is given by $m = \frac{m_0}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)}}$

where m_0 = rest mass of electron = 9×10^{-31} kg.

$$\begin{aligned} \therefore m &= \frac{9 \times 10^{-31}}{\sqrt{\left(1 - \frac{100}{101}\right)}} \left(\text{since } \frac{v^2}{c^2} = \frac{100}{101} \text{ from (1)} \right) \\ &= \frac{9 \times 10^{-31}}{\sqrt{\left(\frac{1}{101}\right)}} \\ &= 9 \times 10^{-31} \times 10.04 = 90.36 \times 10^{-31} \text{ kg} = 9.036 \times 10^{-30} \text{ kg.} \end{aligned}$$

$$E = mc^2 = (9.036 \times 10^{-30} \text{ kg.}) (3 \times 10^8 \text{ m/s})^2 = 8.13 \times 10^{-13} \text{ joules.}$$

Ex. 27. Calculate the kinetic energy of an electron moving with a velocity of 0.98 times the velocity of light in the laboratory system. (Vikram 2005, 1987)

Solution. Relativistic Kinetic energy, $T = (m - m_0) c^2$

where $m = \frac{m_0}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)}}$

Here $v = 0.98c$.

$$m = \frac{m_0}{\sqrt{\left\{1 - \left(\frac{0.98c}{c}\right)^2\right\}}} = \frac{m_0}{\sqrt{\{1 - (0.98)^2\}}} = \frac{m_0}{0.199} = 5.02 m_0$$

$$\therefore \text{Kinetic energy} = (5.02 m_0 - m_0) c^2 = 4.02 m_0 c^2$$

But $m_0 = 9 \times 10^{-31}$ kg., $c = 3 \times 10^8$ m/s

$$\therefore \text{Kinetic energy} = 4.02 \times 9 \times 10^{-31} (3 \times 10^8)^2 = 3.256 \times 10^{-13} \text{ joules.}$$

Ex. 28. An electron and a positron practically at rest come together and annihilate each other, producing two photons of equal energy. Find the energy and equivalent mass of each photon.

Solution. The rest mass of electron = 9×10^{-28} g = 9×10^{-31} kg = rest mass of positron

$\therefore$ The energy of each photon = The energy equivalent to each particle = $m_0 c^2$.

$$= 9 \times 10^{-31} \times (3 \times 10^8)^2 \text{ joules.}$$

$$= 81 \times 10^{-15} \text{ joules} = \frac{81 \times 10^{-15}}{1.6 \times 10^{-19}} \text{ eV.} \approx 5 \times 10^5 \text{ eV}$$

The equivalent mass of each photon = 9×10^{-31} kg.

Ex. 29. Calculate the velocity of electrons accelerated by a potential of 1 million volts.

Solution. We know, the kinetic energy, $T = (m - m_0) c^2$

$$= \left\{ \frac{1}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)}} - 1 \right\} m_0 c^2$$

Given $T = 1 \text{ MeV} = 10^6 \text{ eV} = 10^6 \times 1.6 \times 10^{-19} \text{ joules}$

$m_0 = 9 \times 10^{-31} \text{ kg.}$

$$\therefore 10^6 \times 1.6 \times 10^{-19} = \left[\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \right] \times 9 \times 10^{-31} \times (3 \times 10^8)^2$$

Solving we get

$$v = \frac{2\sqrt{2}}{3} c = \frac{2\sqrt{2}}{2} \times 3 \times 10^8 \text{ m/s.} = 2.2 \times 10^8 \text{ m/s}$$

Ex. 30. What is the annual loss in the mass of the sun, if approximately 2 calories of radiated energy are received by each square cm. of the earth's surface per minute ?

(The distance from the earth to the sun is about $150 \times 10^6 \text{ km.}$)

Solution. Given the distance from the earth to the sun

$$= 150 \times 10^6 \text{ km} = 150 \times 10^9 \text{ m.} = R \text{ (say)}$$

The energy received by each square cm of the earth's surface per minute = 2 calories
 $= 2 \times 4.2 \text{ joules.}$

$\therefore$ The energy received by each square of earth's surface per minute

$$= 2 \times 4.2 \times 10^4 \text{ joules.}$$

So the energy received by each square metre of earth's surface per year

$$= 2 \times 4.2 \times 10^4 \times 60 \times 24 \times 365 \text{ joules.}$$

$\therefore$ Total energy radiated by the sun per year

$$\Delta E = 4\pi R^2 \times 2 \times 4.2 \times 10^4 \times 60 \times 24 \times 365 \text{ joules}$$

$$= 4 \times \pi \times (150 \times 10^9)^2 \times 2 \times 4.2 \times 10^4 \times 60 \times 24 \times 365 \text{ joules.}$$

But according to the mass-energy equivalence we have decrease in mass of sun

$$\Delta m = \frac{\Delta E}{c^2}$$

$$= \frac{4 \times 3.14 \times (150 \times 10^9)^2 \times 2 \times 4.2 \times 10^4 \times 60 \times 24 \times 365}{(3 \times 10^8)^2} \text{ kg per year}$$

$$= 1.387 \times 10^{17} \text{ kg per year.}$$

Ex. 31. A red hot sphere of iron (specific heat of capacity = $0.11 \text{ cal/g. } ^\circ\text{C}$) weighs 1 kg. If the sphere cools to 1200°C , what is the equivalent loss in mass ?

Solution. The heat energy lost by sphere $\Delta E = ms \Delta \theta$

$$= 1000 \times 0.11 \times 1200 \text{ calories} = 1000 \times 0.11 \times 1200 \times 4.2 \text{ joules}$$

$$\Delta m = \frac{\Delta E}{c^2} = \frac{1000 \times 0.11 \times 1200 \times 4.2}{(3 \times 10^8)^2} \text{ kg.} = 5.6 \times 10^{-12} \text{ kg.}$$

Ex. 32. Calculate the wavelength of gamma ray photon produced in two quanta annihilation of an electron and a positron.

($h = 6.6 \times 10^{-34} \text{ joule - sec.}$ $c = 3 \times 10^8 \text{ m/s,}$ $m_e = m_p = 9 \times 10^{-31} \text{ kg.}$) (Kanpur 1995)

Solution. Energy of a photon, $E = h\nu = \frac{hc}{\lambda}$

and Energy of an electron or positron $E = mc^2$ (since $m_e = m_p = m$)

When electron and positron annihilate to produce two photons, we have

$$2mc^2 = \frac{2hc}{\lambda}$$

$$\text{or } \lambda = \frac{hc}{mc^2} = \frac{h}{mc} = \frac{6.6 \times 10^{-34}}{9 \times 10^{-31} \times 3 \times 10^8} \text{ m} = 2.44 \times 10^{-10} \text{ m} = 2.44 \text{ \AA}.$$

Ex. 33. By what fraction does the mass of water increase, due to increase in its thermal energy, when it is heated from 30°C to 80°C ?

Solution. Let mass of water to be heated be M kg.

Energy needed in heating the water from 30°C to 80°C is

$$\Delta E = M \times 1 \times 50 \text{ kilocalories} = M \times 1 \times 50 \times 1000 \times 4.2 \text{ joules} \\ = 2.1 \times 10^5 M \text{ joules.}$$

$$\text{The equivalent mass } \Delta M = \frac{\Delta E}{c^2} = \frac{2.1 \times 10^5 M}{(3 \times 10^8)^2} \text{ kg.} = 2.33 \times 10^{-12} M \text{ kg.}$$

$\therefore$ The fractional increase in mass of water due to heating

$$= \frac{\Delta M}{M} = 2.33 \times 10^{-12} \text{ per kg.}$$

16.17. Relation between Momentum and energy

$$E^2 = p^2 c^2 + m_0^2 c^4.$$

According to mass energy relation, we have

Total energy $E = mc^2$ and momentum $p = mv$

$$\therefore E^2 - p^2 c^2 = (mc^2)^2 - (mv)^2 c^2 = m^2 c^4 - m^2 v^2 c^2.$$

$$\text{But according to mass-velocity relation, } m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

$$\therefore E^2 - p^2 c^2 = \frac{m_0^2}{1 - (v^2/c^2)} \cdot c^4 - \frac{m_0^2}{1 - (v^2/c^2)} \cdot v^2 c^2 \\ = m_0^2 c^4 \left[\frac{1}{1 - (v^2/c^2)} - \frac{(v^2/c^2)}{1 - (v^2/c^2)} \right] = m_0^2 c^4 \cdot \frac{1 - (v^2/c^2)}{1 - (v^2/c^2)} = m_0^2 c^4$$

$$\therefore E^2 = p^2 c^2 + m_0^2 c^4.$$

This is Einstein's well known relation between momentum and energy. This relation may be remembered by right angled triangle of Fig 16-11.

Equation (1) is often written as $E = c\sqrt{(p^2 + m_0^2 c^2)}$.

The energy momentum relation is useful in high energy physics to calculate the energy of a particle when its momentum is known or vice-versa.

Differentiating equation (2) with respect to p we get another useful relation between momentum and energy

$$\frac{dE}{dp} = \frac{pc}{\sqrt{(p^2 + m_0^2 c^2)}} = \frac{pc^2}{E} = \frac{mvc^2}{mc^2} \text{ or } \frac{dE}{dp} = v$$

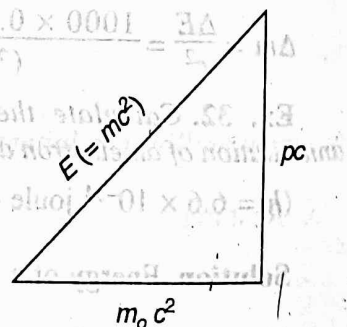


Fig. 16-11

Ex. 34. A π meson of rest mass m_π decays into a μ -meson of mass m_μ and a neutron of mass m_ν . Show that the total energy of the μ -meson is

$$\frac{1}{2m_\pi} [m_\pi^2 + m_\mu^2 - m_\nu^2] c^2. \quad (\text{Pama 2005, Kerala 2002, Kanpur 2004})$$

Solution. According to principle of conservation of momentum, if meson of mass m_μ moves with momentum p , neutrino of mass m_ν moves with momentum $-p$. Therefore according to momentum energy relation

$$E_\mu^2 = p^2 c^2 + m_\mu^2 c^4 \quad \dots(1)$$

$$E_\nu^2 = p^2 c^2 + m_\nu^2 c^4 \quad \dots(2)$$

$$\text{Substrating (2) from (1), we get } E_\mu^2 - E_\nu^2 = (m_\mu^2 - m_\nu^2) c^4 \quad \dots(3)$$

$$\text{Total energy } E_\mu + E_\nu = E = m_\pi c^2 \quad \dots(4)$$

$$\text{Dividing (3) by (4), we get } E_\mu - E_\nu = \frac{m_\mu^2 - m_\nu^2}{m_\pi} c^2$$

Adding (4) and (5), we get

$$2E_\mu = \left(m_\pi + \frac{m_\mu^2 - m_\nu^2}{m_\pi} \right) c^2 = \frac{c^2}{m_\pi} (m_\pi^2 + m_\mu^2 - m_\nu^2)$$

$$\therefore \text{ Energy of } \mu\text{-meson, } E_\mu = \frac{1}{2m_\pi} (m_\pi^2 + m_\mu^2 - m_\nu^2) c^2$$

Ex. 35. A photographic plate shows a track of a cosmic ray particle, placed in a magnetic field, that its energy is 870 MeV and its momentum is 720 MeV/c. What is the mass of the particle? (Bombay 2009, CSE 2004)

Solution. Given $E = 870 \text{ MeV}$

$$p = 720 \text{ MeV/c}$$

The relation between momentum p and energy E is,

$$E^2 = p^2 c^2 + m_0^2 c^4$$

Where m_0 is the rest mass of particle

$$\Rightarrow (m_0 c^2)^2 = E^2 - p^2 c^2$$

$$\Rightarrow m_0 c^2 = \sqrt{E^2 - p^2 c^2}$$

$$\begin{aligned} &= \sqrt{(870 \text{ MeV})^2 - \left(\frac{720 \text{ MeV}}{c} \right)^2 c^2} \\ &= \sqrt{(870)^2 - (720)^2} \text{ MeV} \\ &= 488.4 \text{ MeV} \end{aligned}$$

$$\therefore \text{ Rest mass, } m_0 = 488.4 \frac{\text{MeV}}{c^2}$$

$$\begin{aligned} &= \frac{488.4 \times 1.6 \times 10^{-13}}{(3 \times 10^8)^2} \text{ kg} \\ &= 8.68 \times 10^{-28} \text{ kg} \end{aligned}$$

16.18. Relativistic Doppler's Effect

Consider two systems S and S' , the latter moving with velocity v relative to former along positive x -axis.

Let the transmitter and receiver be situated at origins O and O' of frames S and S' respectively. Let two light signals or pulses be transmitted at time $t = 0$ and $t = T$, T being true period of light pulses. Let $\Delta t'$ be the interval between the reception of these pulses by receiver in S' . Obviously $\Delta t' = T'$ is the observed time interval between these pulses (since observer or receiver is at rest in frame S').

Since observer continues to be at O' all the time, the distance $\Delta x'$ covered by him in frame S' during the reception of two pulses is zero.

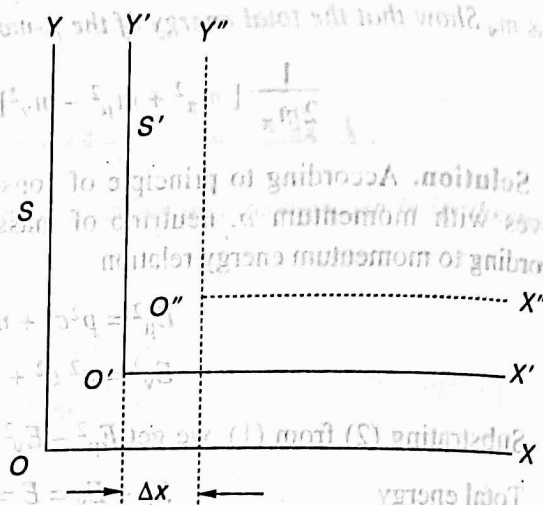


Fig. 16.12

From inverse Lorentz transformations $x = \frac{x' + vt'}{\sqrt{1 - \left(\frac{v^2}{c^2}\right)}}$

we have
$$\Delta x = \frac{\Delta x' + v\Delta t'}{\sqrt{1 - \left(\frac{v^2}{c^2}\right)}} = \frac{v\Delta t'}{\sqrt{1 - \left(\frac{v^2}{c^2}\right)}} \quad (\text{since here } \Delta x' = 0)$$

or
$$\Delta x = \frac{vT'}{\sqrt{1 - \left(\frac{v^2}{c^2}\right)}} \quad (\text{since } \Delta t' = T) \dots (1)$$

This equation shows that the second pulse has to travel this much distance Δx more than the first pulse in frame S along x -axis to be able to reach at origin O' in moving frame S' .

Also from inverse Lorentz transformations $t = \frac{t' + \left(\frac{vx'}{c^2}\right)}{\sqrt{1 - \left(\frac{v^2}{c^2}\right)}}$

we have
$$\Delta t = \frac{\Delta t' + \left(\frac{v\Delta x'}{c^2}\right)}{\sqrt{1 - \left(\frac{v^2}{c^2}\right)}} = \frac{\Delta t'}{\sqrt{1 - \left(\frac{v^2}{c^2}\right)}} \quad (\text{since } \Delta x' = 0)$$

or
$$\Delta t = \frac{T'}{\sqrt{1 - \left(\frac{v^2}{c^2}\right)}} \quad (\text{since } \Delta t' = T) \dots (2)$$

This relation includes both the actual time period T of pulses and the time taken ($\Delta x/c$) by the second pulse to cover the extra distance Δx in frame S . That is

$$\Delta t = T + \frac{\Delta x}{c} \dots (3)$$

Substituting values of Δx and Δt from (1) and (2) in (3), we get

$$\frac{T'}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)}} = T + \frac{vT'}{c \sqrt{\left(1 - \frac{v^2}{c^2}\right)}}$$

on solving

$$T = \frac{T' \left(1 + \frac{v}{c}\right)}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)}} = T' \sqrt{\left(\frac{1 - \frac{v}{c}}{1 + \frac{v}{c}}\right)} \quad \dots(4)$$

If v and v' be the the actual and observed frequencies of light pulses respectively, we have

$$v = \frac{1}{T} \text{ and } v' = \frac{1}{T'}$$

$\therefore$ Equation (4) gives $\frac{1}{v} = \frac{1}{v'} \sqrt{\left(\frac{1 - \frac{v}{c}}{1 + \frac{v}{c}}\right)}$

i.e., $v' = v \sqrt{\left(\frac{1 - \frac{v}{c}}{1 + \frac{v}{c}}\right)} \quad \dots(5)$

This result is known as **Doppler's relativistic formula for light waves in vacuum.**

In terms of actual and apparent wavelengths λ and λ' given by

$$\lambda = \frac{c}{v} \text{ and } \lambda' = \frac{c}{v'}$$

we get

$$\frac{c}{\lambda'} = \frac{c}{\lambda} \sqrt{\left(\frac{1 - \frac{v}{c}}{1 + \frac{v}{c}}\right)}$$

i.e.,

$$\lambda' = \lambda \sqrt{\left(\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}\right)} \quad \dots(6)$$

In both relations (5) and (6) v is positive or negative according as the observer is receding from or approaching the source of light, so that during recession the apparent frequency of pulse v' decreases and its apparent wavelength λ' increases and during approach the apparent frequency v' increases and the apparent wavelength λ' decreases. The Doppler effect so far considered is the **longitudinal Doppler's effect** since the observations are made along the direction of travel of light source.

If the observations are made at right angles to the direction of travel of light source, the Doppler effect observed is known as **transverse Doppler's effect**. This is found usually in the case of atoms. Unlike the classical theory, the theory of relativity predicts **transverse Doppler's effect**.

For transverse Doppler's effect the second signal does not take on extra time since here $y = y'$ and $z = z'$ due to no relative motion along y and z -axis.

Therefore, from the results of time dilation, the real and observed periods are related by

$$T' = \frac{T}{\sqrt{1 - \frac{v^2}{c^2}}} \quad \dots(7)$$

If ν and ν' are the actual and observed frequencies in S and S' respectively, then $T' = \frac{1}{\nu'}$ and $T = \frac{1}{\nu}$ so that equation (7) then gives

$$\nu' = \nu \sqrt{1 - \frac{v^2}{c^2}} \quad \dots(8)$$

This expression represents **transverse Doppler's effect**. If $v \ll c$, the longitudinal and transverse Doppler's effect take the form

$$\begin{aligned} \nu' &= \nu \sqrt{1 - \frac{v^2}{c^2}} && \text{(longitudinal Doppler's effect)} \\ \nu' &= \nu && \text{(transverse Doppler effect)} \end{aligned}$$

Then according to the classical theory, there is no transverse Doppler Effect; but according to the theory of relativity transverse Doppler's change in frequency is given by equation (8).

Confirmation of Doppler's Effect : The longitudinal Doppler Effect was confirmed by H.E. lives and G.R Stilwell in 1941 spectroscopically using beams of hydrogen atoms in excited electronic states Atomic hydrogen was produced by the result of breaking up accelerated molecular hydrogen ions using intense electric field.

They observed the shift in the average wavelength of a particular spectral line of hydrogen atoms. The shift was observed by taking the average over forward and backward directions relative to the line of flight of atoms.

The longitudinal Doppler's effect is expressed as $\lambda' = \lambda \sqrt{\frac{1 \pm \frac{v}{c}}{1 \mp \frac{v}{c}}}$

If v is (+) ve in forward direction, it is (-)ve in backward direction, thus if λ_1 is wavelength observed in forward direction, λ_0 the wavelength emitted by atom at rest, then from (9)

$$\lambda_1 = \lambda_0 \sqrt{\frac{1 + \left(\frac{v}{c}\right)}{1 - \left(\frac{v}{c}\right)}}$$

If λ_2 is wavelength observed in the backward direction, we have

$$\lambda_2 = \lambda_0 \sqrt{\frac{1 - \frac{v}{c}}{1 + \frac{v}{c}}}$$

Therefore, the average observed wavelength is given by

$$\lambda = \frac{\lambda_1 + \lambda_2}{2} = \frac{\lambda_0}{2}$$

$$\left[\sqrt{\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}} + \sqrt{\frac{1 - \frac{v}{c}}{1 + \frac{v}{c}}} \right] = \frac{\lambda_0}{\sqrt{1 - \frac{v^2}{c^2}}} \quad \dots(10)$$

The shift observed by them was 0.074×10^{-8} cm while shift calculated from (10) taking value of $\frac{v}{c} = 0.005$ is 0.072×10^{-8} cm. Thus observed and calculated values very nearly agree, thereby verifying longitudinal Doppler's Effect.

Ex. 36. Calculate the wavelength shift in the relativistic Doppler effect for the H_α (6563 Å) line emitted by a star receding from the earth with a relative velocity $0.1 c$. Is the classical (first order) result a good approximation?

Solution. The new wavelength λ' is given by $\lambda' = \lambda \sqrt{\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}}$

$$= 6563 \sqrt{\frac{1 + 0.1}{1 - 0.1}} \quad \left(\text{since } \frac{v}{c} = 0.1 \right) = 6563 \sqrt{\frac{1.1}{0.9}} = 7256 \text{ Å}$$

$\therefore$ The wavelength shift, $\lambda' - \lambda = (7256 - 6563) \text{ Å} = 693 \text{ Å}$.

Now looking for the classical result, we have

$$\lambda' = \lambda \sqrt{\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}} = \lambda \left(1 + \frac{v}{c} \right)^{1/2} \left(1 - \frac{v}{c} \right)^{-1/2} = \lambda \left(1 + \frac{v}{2c} \right) \left(1 + \frac{v}{2c} \right) \approx \lambda \left(1 + \frac{v}{c} \right)$$

in classical limit $\lambda' = \lambda (1 + 0.1)$

or $\lambda' - \lambda = 0.1 \lambda = 0.1 \times 6563 \text{ Å} = 656.3 \text{ Å}$.

The deviation of classical result from relativistic result $= \frac{693 - 656.3}{693} \times 100\% = 5.3\%$

The deviation depends on ratio v/c .

16-19. Phenomenon of Aberration (Relativistic)

The phenomenon of aberration originally discovered by Bradley in 1727 is very useful to determine the velocity of earth when the velocity of light is known. According to Bradley all the stars except those at the zenith i.e., directly overhead move in elliptical orbits with a period of one year, while those at the zenith move in a circular orbit. Therefore a seasonal change in the position of stars is observed. The phenomenon of aberration results "The speed of light is independent of the medium of transmission; but the direction of light rays depends on the motion of the source emitting light relative to the observer."

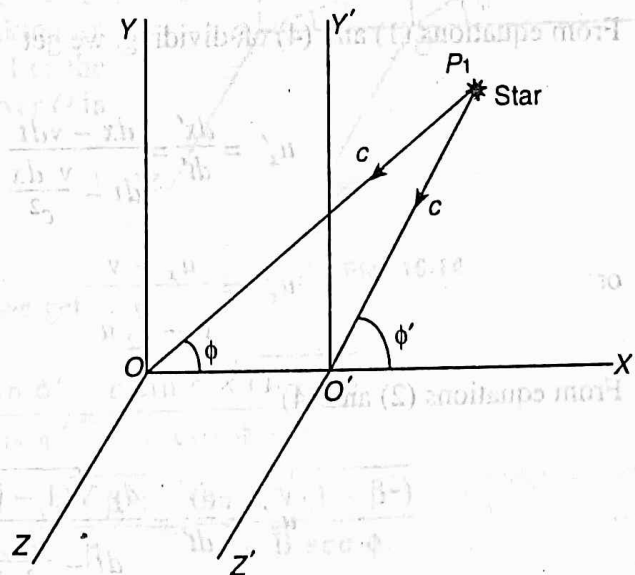


Fig. 16-13

As we know that the earth revolves round the sun in its orbit, therefore we may imagine the sun to be system S while the earth as system S' which is moving with velocity v relative to system S along (+)ve direction of common X -axis. Let the light from a star P be observed by observers O and O' in systems S and S' respectively.

Let the angles made by a light ray in $X-Y$ plane from the star P at any instant in two systems at O and O' be ϕ and ϕ' respectively. If any particle P has velocity $\mathbf{u}$ in system S and $\mathbf{u}'$ in system S' , then we have

$$\mathbf{u} = \hat{i}u_x + \hat{j}u_y + \hat{k}u_z$$

$$\mathbf{u}' = \hat{i}'u'_x + \hat{j}'u'_y + \hat{k}'u'_z$$

where u_x, u_y , and u_z are components of $\mathbf{u}$ along three axes in system S and $\hat{i}, \hat{j}, \hat{k}$ are unit vectors along these axes.

Dashes represent the same quantities for system S' .

Now according to Lorentz transformation equations, we have

$$x' = \frac{x - vt}{\sqrt{1 - \beta^2}}$$

$$y' = y$$

$$z' = z$$

$$t' = \frac{t - \frac{vx}{c^2}}{\sqrt{1 - \beta^2}}$$

$$\text{where } \beta = \frac{v}{c}$$

Taking differentials we have

$$dx' = \frac{dx - vdt}{\sqrt{1 - \beta^2}} \quad \dots(1)$$

$$dy' = dy \quad \dots(2)$$

$$dz' = dz$$

$$dt' = \frac{dt - \frac{vdx}{c^2}}{\sqrt{1 - \beta^2}} \quad \dots(4)$$

From equations (1) and (4) on dividing, we get

$$u'_x = \frac{dx'}{dt'} = \frac{dx - vdt}{dt - \frac{v}{c^2} dx} = \frac{\frac{dx}{dt} - v}{1 - \frac{v}{c^2} \frac{dx}{dt}}$$

or

$$u'_x = \frac{u_x - v}{1 - \frac{v}{c^2} u_x}$$

From equations (2) and (4)

$$u'_y = \frac{dy'}{dt'} = \frac{dy \sqrt{1 - \beta^2}}{dt - \frac{v}{c^2} dx} = \frac{\frac{dy}{dt} \sqrt{1 - \beta^2}}{1 - \frac{v}{c^2} \frac{dx}{dt}}$$

$$= \frac{u_y \sqrt{1 - \beta^2}}{1 - \frac{v}{c^2} u_x} \quad \dots(6)$$

Now from figure 16.13 it is clear that the star light travelling in $x-y$ plane with velocity c , has components, $c \cos(\pi + \phi)$ and $c \cos(\pi + \phi')$ along (+)ve direction of X -axis in system, S and S' respectively. Also those $+c \sin(\pi + \phi)$ and $+c \sin(\pi + \phi')$ along (+)ve direction of y -axis in systems S and S' respectively. Thus we have

$$\left. \begin{aligned} u_x &= +c \cos(\pi + \phi) = -c \cos \phi, & u_x' &= +c \cos(\pi + \phi') = -c \cos \phi' \\ u_y &= +c \sin(\pi + \phi) = -c \sin \phi, & u_y' &= +c \sin(\pi + \phi') = -c \sin \phi' \end{aligned} \right\} \dots(7)$$

Now from (5) and (6), we have

$$\frac{u_y'}{u_x'} = \frac{1 - \frac{v}{c^2} u_x}{\frac{u_y \sqrt{1 - \beta^2}}{1 - \frac{v}{c^2} u_x}} = \frac{u_y \sqrt{1 - \beta^2}}{u_x - v} \quad \dots(8)$$

Substituting values from (7)

$$\frac{u_y'}{u_x'} = \frac{-c \sin \phi'}{-c \sin \phi} = \frac{-c \sin \phi \sqrt{1 - \beta^2}}{-c \cos \phi - v}$$

or

$$\tan \phi' = \frac{\sin \phi \sqrt{1 - \beta^2}}{\cos \phi + \beta} = \frac{\tan \phi \sqrt{1 - \beta^2}}{1 + \beta \sec \phi} \quad \dots(9)$$

The inverse of equation (9) can be written at once as

$$\tan \phi = \frac{\sin \phi' \sqrt{1 - \beta^2}}{\cos \phi' - \beta} = \frac{\tan \phi' \sqrt{1 - \beta^2}}{1 - \beta \sec \phi'} \quad \dots(10)$$

Instead of light coming from the star, consider now that the light signal is emitted from a source at the origin of system S' as shown in Fig. 16.14. Consider a light ray in $X'-Y'$ plane making an angle ϕ' with the X' -axis in system S' . Let the corresponding angle as measured by observer O in system S be ϕ . Then in this case

$$u_x = c \cos \phi, \quad u_x' = c \cos \phi'$$

$$u_y = c \sin \phi, \quad u_y' = c \sin \phi'$$

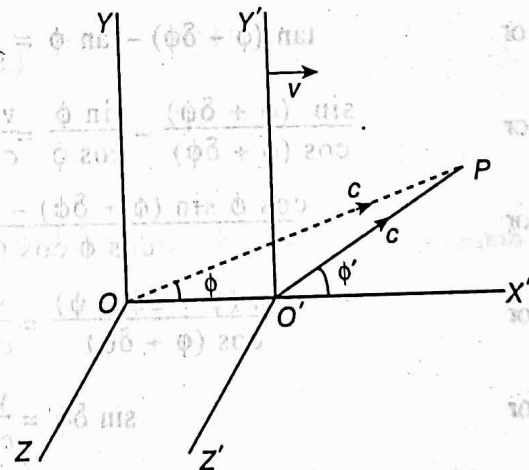


Fig. 16.14

Substituting these values in eqn. (8), we get

$$\frac{u_y'}{u_x'} = \frac{-c \sin \phi'}{-c \cos \phi'} = \frac{c \sin \phi \sqrt{1 - \beta^2}}{c \cos \phi - v}$$

$$\therefore \tan \phi' = \frac{\sin \phi \sqrt{1 - \beta^2}}{\cos \phi - \beta} = \frac{\tan \phi \sqrt{1 - \beta^2}}{1 - \beta \sec \phi} \quad \dots(11)$$

The inverse of equation (11) can be written as

$$\tan \phi = \frac{\sin \phi' \sqrt{1 + \beta^2}}{\cos \phi' + \beta} = \frac{\tan \phi' \sqrt{1 - \beta^2}}{1 + \beta \sec \phi'} \quad \dots(12)$$

From equations (9), (10), (11) and (12) it is obvious that ϕ and ϕ' are not the same in the two systems; i.e., the direction of light rays depends upon relative motion of source and observer; therefore the explanation of the phenomenon of aberration is obvious. Any of equations (9) to (12) gives the *exact relativistic aberration formula*.

Classical formula for aberration from relativistic formula

Now from equation (11), we have

$$\tan \phi' = \tan \phi \sqrt{\left(1 - \frac{v^2}{c^2}\right) \left(1 - \frac{v}{c \cos \phi}\right)^{-1}}$$

If $v \ll c$, then the terms of containing $\frac{v^2}{c^2}$ and higher powers of $\frac{v}{c}$ can be neglected; whence we have

$$\tan \phi' = \tan \phi \left(1 + \frac{v}{c \cos \phi}\right) \quad \dots(13)$$

This formula contains only term containing $\frac{v}{c}$ and it is the classical formula for aberration.

Now putting $\phi' = \phi + \delta\phi$, we have from equation (13);

$$\tan(\phi + \delta\phi) = \tan \phi + \frac{v \tan \phi}{c \cos \phi}$$

$$= \tan \phi + \frac{v \sin \phi}{c (\cos \phi)^2}$$

or $\tan(\phi + \delta\phi) - \tan \phi = \frac{v \sin \phi}{c \cos^2 \phi}$

or $\frac{\sin(\phi + \delta\phi)}{\cos(\phi + \delta\phi)} - \frac{\sin \phi}{\cos \phi} = \frac{v \sin \phi}{c \cos^2 \phi}$

or $\frac{\cos \phi \sin(\phi + \delta\phi) - \sin \phi \cos(\phi + \delta\phi)}{\cos \phi \cos(\phi + \delta\phi)} = \frac{v \sin \phi}{c \cos^2 \phi}$

or $\frac{\sin(\phi + \delta\phi - \phi)}{\cos(\phi + \delta\phi)} = \frac{v \sin \phi}{c \cos \phi}$

or $\sin \delta\phi = \frac{v \sin \phi}{c \cos \phi} (\cos \phi + \delta\phi)$

Now since $\delta\phi$ is very small, we may put

$$\sin \delta\phi = \delta\phi \quad \text{and} \quad \cos(\phi + \delta\phi) = \cos \phi,$$

thus we have

$$\delta\phi = \frac{v}{c} \sin \phi$$

which is classical formula for aberration giving maximum value for aberration classically

16-20. Relativistic Lagrangian and Hamiltonian

1. Relativistic Lagrangian

Consider a particle of rest mass m_0 moving with velocity $\mathbf{v} = \dot{x}_1 \hat{i} + \dot{x}_2 \hat{j} + \dot{x}_3 \hat{k}$ in a field of potential V . As in non-relativistic case, assuming that the momentum p_i corresponding to generalised coordinate x_i is

$$p_i = \frac{\partial L}{\partial \dot{x}_i} \quad \dots(1)$$

We have

$$p_i = m v_i = \frac{m_0 \dot{x}_i}{\sqrt{1 - \frac{v^2}{c^2}}}$$

But

$$v^2 = \sum_i \dot{x}_i^2$$

$\therefore$

$$p_i = \frac{m_0 \dot{x}_i}{\sqrt{1 - \frac{\sum_i \dot{x}_i^2}{c^2}}}$$

Substituting this value in equation (1), we get

$$\frac{m_0 \dot{x}_i}{\sqrt{1 - \frac{\sum_i \dot{x}_i^2}{c^2}}} = \frac{\partial L}{\partial \dot{x}_i} \quad \dots(2)$$

Integrating with respect to $\dot{x}_i$, we get

$$L = \int \frac{m_0 \dot{x}_i d\dot{x}_i}{\sqrt{1 - \frac{\sum_i \dot{x}_i^2}{c^2}}} - V(x_i) \quad \dots(3)$$

where $-V(x_i)$ is constant of integration. It has been written keeping non-relativistic expression for Lagrangian and the introduce the potential function.

Substituting $1 - \frac{\sum_i \dot{x}_i^2}{c^2} = t$, we have

$$-\frac{2\dot{x}_i d\dot{x}_i}{c^2} = dt \Rightarrow \dot{x}_i d\dot{x}_i = -\frac{c^2}{2} dt$$

$$\begin{aligned} L &= -\frac{m_0 c^2}{2} \int t^{-1/2} dt - V(x_i) \\ &= -\frac{m_0 c^2}{2} t^{1/2} - V(x_i) \end{aligned}$$

$$= -m_0 c^2 \sqrt{1 - \frac{v^2}{c^2}} - V(x_{ii}) \quad \dots(4)$$

= Relativistic K.E. + Potential energy.

The correctness of Lagrangian can be seen by using Lagrange's equation.

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}_i} \right) - \frac{\partial L}{\partial x_i} = 0, \quad i = 1, 2, 3$$

From (4),
$$\frac{\partial L}{\partial x_i} = -\frac{\partial V}{\partial x_i}$$

$$\begin{aligned} \frac{\partial L}{\partial \dot{x}_i} &= \frac{\partial}{\partial \dot{x}_i} \left[-m_0 c^2 \left\{ 1 - \frac{\sum \dot{x}_i^2}{c^2} \right\}^{1/2} - V(x_i) \right] \\ &= \frac{m_0 \dot{x}_i}{\sqrt{1 - \frac{\sum \dot{x}_i^2}{c^2}}} = \frac{m_0 \dot{x}_i}{\sqrt{1 - \frac{v^2}{c^2}}} \end{aligned}$$

So Lagrange's equation gives

$$\frac{d}{dt} \left(\frac{m_0 \dot{x}_i}{\sqrt{1 - \frac{v^2}{c^2}}} \right) + \frac{\partial V}{\partial x_i} = 0$$

$$\Rightarrow \frac{d}{dt} \left(\frac{m_0 v_i}{\sqrt{1 - \frac{v^2}{c^2}}} \right) = -\frac{\partial V}{\partial x_i} = F_i$$

which agrees with Newton's equation of motion

$$F_i = \frac{dp_i}{dt} = \frac{d}{dt} \left(\frac{m_0 v_i}{\sqrt{1 - \frac{v^2}{c^2}}} \right),$$

thereby indicating that the Lagrangian is correct.

Expression for relativistic kinetic energy is

$$T = (m - m_0) c^2 = \frac{m_0 c^2}{\sqrt{1 - \frac{v^2}{c^2}}} - m_0 c^2 \quad \dots (5)$$

So in analogy with *classical Lagrangian*, the Lagrangian should be

$$L = T - V = \left(\frac{m_0 c^2}{\sqrt{1 - \frac{v^2}{c^2}}} - m_0 c^2 \right) - V(x_i) \quad \dots (6)$$

A comparison of (4) and (6) shows that the relativistic Lagrangian is no longer equal to $T - V$, but the partial derivative of Lagrangian with respect to generalised velocity is equal to generalised momentum. This insures the correctness of Lagrangian given by equation (4).

The Lagrangian analogous to *Classical Lagrangian* may be obtained by adding rest mass energy $m_0 c^2$, i.e.,

$$L_R = L + m_0 c^2 = m_0 c^2 \left\{ 1 - \sqrt{1 - \frac{v^2}{c^2}} \right\} - V(x_i) \quad \dots (7)$$

2. Relativistic Hamiltonian

The relativistic Hamiltonian function H is defined as

Substituting value of L_C from (7)

...(8)

$$H = \sum \left[\frac{\partial}{\partial \dot{x}} \left\{ m_0 c^2 \left(1 - \sqrt{1 - \frac{v^2}{c^2}} \right) - V \right\} \right] \dot{x} - m_0 c^2 \left\{ 1 - \sqrt{1 - \frac{v^2}{c^2}} \right\} + V(x_i)$$

$$= m_0 c^2 \sum \frac{\partial}{\partial \dot{x}} \left\{ 1 - \sqrt{1 - \frac{\sum \dot{x}_i^2}{c^2}} \right\} \dot{x} - m_0 c^2 \left\{ 1 - \sqrt{1 - \frac{\sum \dot{x}_i^2}{c^2}} \right\} + V(x_i)$$

$$= \sum \frac{m_0 \dot{x}_i^2}{\sqrt{1 - \frac{v^2}{c^2}}} - m_0 c^2 \left\{ 1 - \sqrt{1 - \frac{v^2}{c^2}} \right\} + V(x_i)$$

$$= \frac{m_0 \sum \dot{x}_i^2}{\sqrt{1 - \frac{v^2}{c^2}}} - m_0 c^2 \left\{ 1 - \sqrt{1 - \frac{v^2}{c^2}} \right\} + V(x_i)$$

$$= \frac{m_0 v^2}{\sqrt{1 - \frac{v^2}{c^2}}} - m_0 c^2 \left\{ 1 - \sqrt{1 - \frac{v^2}{c^2}} \right\} + V(x_i)$$

$$= m_0 c^2 \left[\frac{\frac{v^2}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} - 1 + \sqrt{1 - \frac{v^2}{c^2}} \right] + V(x_i)$$

$$= m_0 c^2 \left[\frac{\frac{v^2}{c^2} + 1 - \frac{v^2}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} - 1 \right] + V(x_i)$$

$$= m_0 c^2 \left[\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} - 1 \right] + V(x_i) \quad (9)$$

This is relativistic Hamiltonian function.

Clearly $H = T + V$

= Relativistic K.E. + Potential energy.

Thus like classical Hamiltonian, relativistic Hamiltonian also represents the total energy of the system.

(8)...

$$p_x = \frac{m_0 \dot{x}_i}{\sqrt{1 - \frac{v^2}{c^2}}} \quad i = 1, 2, 3$$

∴

$$p^2 = \sum_i p_i^2 = \frac{m_0^2}{1 - \frac{v^2}{c^2}} \sum_i \dot{x}_i^2$$

$$= \frac{m_0^2 v^2}{1 - \frac{v^2}{c^2}} = -m_0^2 c^2 + \frac{m_0^2 c^2}{1 - \frac{v^2}{c^2}}$$

or

$$p^2 + m_0^2 c^2 = \frac{m_0^2 c^2}{1 - \frac{v^2}{c^2}}$$

or

$$\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \sqrt{\frac{p^2 + m_0^2 c^2}{m_0^2 c^2}} \quad \dots(10)$$

$$H = c \sqrt{p^2 + m_0^2 c^2} - m_0 c^2 + V$$

Substituting this value of $\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$ in equation(9), the expression for the

relativistic Hamiltonian becomes

$$H = m_0 c^2 \left[\sqrt{\left\{ \frac{p^2 + m_0^2 c^2}{m_0^2 c^2} \right\}} - 1 \right] + V$$

or

$$H = [c \sqrt{p_x^2 + p_y^2 + p_z^2 + m_0^2 c^2}] - m_0 c^2 + V \quad \dots(11)$$

This is the required expression for the relativistic Hamiltonian.

Note: For a free particle $V=0$, hence for a free particle, the Lagrangian and Hamiltonian take the form

$$L = m_0 c^2 \left[1 - \left(1 - \frac{v^2}{c^2} \right)^{1/2} \right]$$

...(12)

$$H = c \sqrt{p_x^2 + p_y^2 + p_z^2 + m_0^2 c^2} - m_0 c^2$$

...(13)

16.21. Minkowski's Space

Minkowski pointed out that the external world is not formed of ordinary three dimensional space, known as *Euclidean space*, but it is four dimensional space-time continuum known as *Minkowski space* or *world space*, where time or more conveniently *ict* may be regarded to be fourth dimension. Therefore, an event in world space must be represented by four coordinates (x_1, x_2, x_3, x_4) out of which first three are space coordinates and the fourth one is time

coordinate. The events in the world are represented by points known as *world points*. In this world space there corresponds to each particle a certain line known as *world line*. Let us consider two axes OX and OP , O being origin of the system (where $p = ict$).

We know that according to Lorentz transformations

$$x^2 + y^2 + z^2 - c^2 t^2 = \text{invariant}$$

but

$$y' = y, z' = z$$

$$x^2 - c^2 t^2 = \text{invariant or } x^2 + p^2 = \text{invariant}$$

This means that the distance of any variable point P in x - p plane from origin is unchanged. We shall now show that a rotation of the (x, p) system of axes gives Lorentz transformations.

Let the axes OX and OP be rotated through an angle θ , such that the new positions of OX and OP are OX' and OP' , then we have the radius vector in both the systems

$$r = \hat{i}x + \hat{p}p = \hat{i}'x' + \hat{p}'p', \quad \dots(1)$$

where $\hat{i}, \hat{p}, \hat{i}'$ and $\hat{p}'$ are unit vectors along x, p, x' and p' respectively.

From Fig. 16-15, we have

$$\text{and } \left. \begin{aligned} \hat{i} &= \hat{i}' \cos \theta - \hat{p}' \sin \theta \\ \hat{p} &= \hat{p}' \cos \theta + \hat{i}' \sin \theta \end{aligned} \right\}$$

Substituting these values in equations (1) we get

$$(\hat{i}' \cos \theta - \hat{p}' \sin \theta)x + (\hat{p}' \cos \theta + \hat{i}' \sin \theta)p = \hat{i}'x' + \hat{p}'p'$$

Equating coefficients of $\hat{i}'$ and $\hat{p}'$ on either side, we get

$$\left. \begin{aligned} x' &= x \cos \theta + p \sin \theta \quad \dots(i) \\ p' &= -x \sin \theta + p \cos \theta \quad \dots(ii) \end{aligned} \right\} \quad \dots(3)$$

Let us substitute,

$$\tan \theta = \frac{iv}{c} = i\beta$$

$$\Rightarrow \sin \theta = \frac{i\beta}{\sqrt{1 - \beta^2}} \text{ and } \cos \theta = \frac{1}{\sqrt{1 - \beta^2}}$$

in equation (3), we get from (3)

$$x' = \frac{x}{\sqrt{1 - \beta^2}} + \frac{pi\beta}{\sqrt{1 - \beta^2}} = \frac{x + i\beta p}{\sqrt{1 - \beta^2}}$$

$$= \frac{x + \frac{iv}{c}(ict)}{\sqrt{1 - \beta^2}}$$

$$\Rightarrow x' = \frac{x - vt}{\sqrt{1 - \beta^2}} \quad \dots(4)$$

Equation [3 (ii)] gives

$$p' = \frac{-xi\beta}{\sqrt{1 - \beta^2}} + \frac{p}{\sqrt{1 - \beta^2}} = \frac{p - ix\beta}{\sqrt{1 - \beta^2}}$$

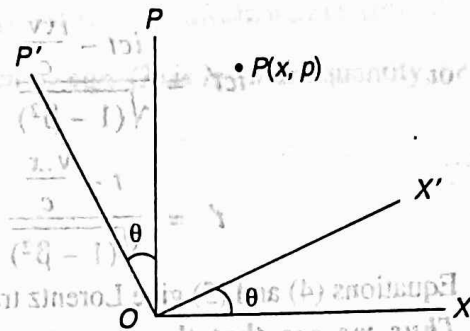


Fig. 15-15

$$\text{or } \begin{aligned} ict' &= \frac{ict - \frac{ixv}{c}}{\sqrt{1 - \beta^2}} \\ t' &= \frac{t - \frac{v \cdot x}{c}}{\sqrt{1 - \beta^2}} \end{aligned} \quad (5)$$

Equations (4) and (5) give Lorentz transformations.

Thus we see that the Lorentz transformations are equivalent to rotation of axes in four dimensional space (x, y, z, p) through a hypothetical angle $i\beta$.

This makes possible to extend ordinary vector analysis to four dimensions, to derive generally valid laws in the form of equations between four dimensional vectors.

16-22. Four Vectors

Having introduced the idea of four dimensional space, it is possible to extend ordinary vector analysis to four dimensions to derive generally valid laws in the form of equations between four dimensional vector. These four-dimensional vectors are called **four vectors** or **world vectors**.

As the four-dimensional coordinates are orthogonal, we must have

$$\hat{i} \cdot \hat{i} = 1, \hat{j} \cdot \hat{j} = 1, \hat{k} \cdot \hat{k} = 1, \hat{p} \cdot \hat{p} = 0$$

and

$$\hat{i} \cdot \hat{j} = 0, \hat{i} \cdot \hat{k} = 0, \hat{i} \cdot \hat{p} = 0, \hat{j} \cdot \hat{p} = 0, \hat{k} \cdot \hat{p} = 0$$

and so on. The world vectors are denoted by bars beneath their symbols.

If $\underline{A}$ and $\underline{B}$ are two world vectors, then

$$\underline{A} = \hat{i} A_x + \hat{j} A_y + \hat{k} A_z + \hat{p} A_p$$

$$\underline{B} = \hat{i} B_x + \hat{j} B_y + \hat{k} B_z + \hat{p} B_p$$

The transformations law in Minkowski four dimensional space for the set of four coordinates (x_1, x_2, x_3, p) in one frame of reference to the set of four co-ordinates (x_1', x_2', x_3', p') in another frame moving with velocity v relative to the first frame is given by

$$x_\mu' = \sum_v a_{\mu v} x_v$$

with the constraint

$$x_1^2 + x_2^2 + x_3^2 + p^2 = x_1'^2 + x_2'^2 + x_3'^2 + p'^2 \quad \dots(1)$$

Any set of four components A_μ which transform under Lorentz transformation like the four-components (x_1, x_2, x_3, p) is called a **four vector**. The components A_μ of four vector $\underline{A}$ transform under Lorentz transformation according to (with $A_0 = iA_4$)

$$A_\mu' = \sum_{\nu=1}^3 a_{\mu\nu} A_\nu$$

the quantities $a_{\mu\nu}$ being given by eqn. (1).

The scalar product of two world vectors $\underline{A}$ and $\underline{B}$ is defined as

$$\underline{A} \cdot \underline{B} = A_x B_x + A_y B_y + A_z B_z + A_p B_p$$

which may also be written as

$$\sum_\mu A_\mu B_\mu = A_1 B_1 + A_2 B_2 + A_3 B_3 + A_0 B_0$$

$$= A_1 B_1 + A_2 B_2 + A_3 B_3 - A_4 B_4 = \underline{A} \cdot \underline{B} - A_4 B_4 \quad \dots(3)$$

where $A_\mu = (A_1, A_2, A_3, A_0)$ or (A_1, A_2, A_3, iA_4)

It can easily be verified that the scalar product given by eqn. (3) is invariant quantity, i.e.,

$$\sum_\mu A'_\mu B'_\mu = \sum_\lambda A_\lambda B_\lambda = \text{constant} \quad \dots(4)$$

$$\begin{aligned} \text{Since } \sum_\mu A'_\mu B'_\mu &= \sum_\mu \sum_\lambda \sum_\nu a_{\mu\lambda} A_\lambda a_{\mu\nu} B_\nu \\ &= \sum_\lambda \sum_\nu \left(\sum_\mu a_{\mu\lambda} a_{\mu\nu} \right) A_\lambda B_\nu \\ &= \sum_\lambda \sum_\nu \delta_{\lambda\nu} A_\lambda B_\nu \quad \left(\text{as } \sum_\mu a_{\mu\nu} a_{\mu\lambda} = \delta_{\lambda\nu} \right) \\ &= \sum_\lambda A_\lambda B_\lambda \quad \left(\text{Here } \delta_{\lambda\nu} \text{ is Kronecker delta symbol} \right) \end{aligned}$$

i.e., the scalar product of two four vectors is invariant quantity.

The definition of vector product of two world vectors is somewhat difficult, because the combination forms an antisymmetric tensor in three dimensions which is represented by a vector. In four dimensions an antisymmetric tensor has six components while a vector has only four.

The vector product of two four vectors $\underline{A}$ and $\underline{B}$ is given by

$$\underline{A} \times \underline{B} = \begin{vmatrix} A_1 & A_2 & A_3 & A_4 \\ B_1 & B_2 & B_3 & B_4 \end{vmatrix}$$

Corresponding to three dimensional ∇ -operator (del operator) we have a four dimensional operator $\square$ operator (D' Alembertian) defined as

$$\square = i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} + \hat{p} \frac{\partial}{\partial p} \quad \dots(5.35)$$

Thus in four-dimensions, we have

$$\underline{\text{grad}} \underline{A} = \square \underline{A} = i \frac{\partial A_x}{\partial x} + j \frac{\partial A_y}{\partial y} + k \frac{\partial A_z}{\partial z} + \hat{p} \frac{\partial A_p}{\partial p} \quad \dots(5.36)$$

$$\underline{\text{div}} \underline{A} = \square \underline{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} + \frac{\partial A_p}{\partial p} \quad \dots(5.37)$$

$$\underline{\text{Curl}} \underline{A} = \square \times \underline{A} = \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} & \frac{\partial}{\partial p} \\ A_x & A_y & A_z & A_p \end{bmatrix} \quad \dots(5.38)$$

As one of the component of any four-vector is imaginary, the square of four-vectors may be (+)ve or (-)ve unlike ordinary vectors. If the square of magnitude of a four vector is greater than or equal to zero, the four-vector is said to be **space-like**; but if the magnitude is negative the four-vector is said to be **time like**. The position vector of a point in four-dimensional space is given by

$$\underline{r} = i x + j y + k z + \hat{p} p \quad \dots(5.39)$$

Let us now consider a difference vector between two world points which may be either (+)ve or (-)ve. Let $\underline{R}$ be the difference vector defined as

$$\underline{R} = \underline{r}_2 - \underline{r}_1$$

where $\underline{r}_1$ and $\underline{r}_2$ are the position vectors of two world points respectively.

Then the magnitude of $\underline{R}$ is given by

$$\underline{R}^2 = |\underline{r}_2 - \underline{r}_1|^2 = |\underline{r}_2 - \underline{r}_1|^2 + (p_2 - p_1)^2$$

where $\underline{r}_1$ and $\underline{r}_2$ are ordinary position vectors.

But

$$p = ict, \text{ i.e., } p_1 = ict_1 \text{ and } p_2 = ict_2$$

$\therefore$

$$\underline{R}^2 = |\underline{r}_2 - \underline{r}_1|^2 - c^2 (t_2 - t_1)^2 \quad \dots(5.40)$$

Thus $\underline{R}$ will be space like if

$$|\underline{r}_2 - \underline{r}_1|^2 > c^2 (t_2 - t_1)^2 \quad \dots(5.41)$$

and $\underline{R}$ will be time like if

$$|\underline{r}_2 - \underline{r}_1|^2 < c^2 (t_2 - t_1)^2 \quad \dots(5.42)$$

Let the spatial axes be so oriented that the difference vector $(\underline{r}_2 - \underline{r}_1)$ is along x -axis and therefore fourth component of $\underline{R}$ may be written as

$$c(t_2' - t_1') = \frac{c(t_2 - t_1) - \frac{v}{c}(x_2 - x_1)}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\sqrt{1 - \frac{v^2}{c^2}}$$

$$t - \frac{vx}{c^2}$$

$$\text{like } t' =$$

$$\sqrt{1 - \frac{v^2}{c^2}}$$

If $\underline{R}$ is space like, then $c(t_2 - t_1) < x_2 - x_1$.

Therefore it is possible to find a velocity $v \ll c$ such that $ic(t_2' - t_1')$ vanishes. vanishing of $ic(t_2' - t_1')$, i.e., fourth component of $\underline{R}$, means that if the distance between two events is space-like it is possible to find reference system in which the two events are simultaneous.

Some examples of four vectors : Let us consider the motion of the material particle in Minkowski (3 + 1) dimensional space. An arbitrary motion of a material particle in Minkowski space can be described by equations of the form

$$x_\mu = x_\mu(x_4) \quad (\mu = 1, 2, 3, \dots) \quad \dots(5.43)$$

where $x_\mu(x_4)$ are definite functions of time variable x_4 .

Equation (43) represents a curve in (3 + 1) dimensional space which is generally known as *time track of the particle*. In the case of a uniform motion the functions $x_\mu(x_4)$ are linear and the time track is a straight line.

In relativistic mechanics time t is not absolute, but the proper time τ is invariant. The proper time of a particle is a measure of the length of the time track. If u is the velocity of a particle, the proper time interval $\Delta\tau$ is given by

$$\Delta\tau = \Delta t \sqrt{1 - \frac{u^2}{c^2}} \quad \dots(5.44)$$

Velocity four-vector or four-velocity : The components of velocity four vector or four velocity are given by

$$u_i = \frac{dx_i}{d\tau}$$

where dx_i are the co-ordinate increments on the time track of a particle.

The components of velocity four vector $\underline{u}$ are written as

$$\left. \begin{aligned} u_1 &= \frac{dx_1}{d\tau} = \frac{dx_1}{dt} \cdot \frac{dt}{d\tau} = \frac{u_x}{\sqrt{1 - \frac{u^2}{c^2}}} \\ u_2 &= \frac{dx_2}{d\tau} = \frac{dx_2}{dt} \cdot \frac{dt}{d\tau} = \frac{u_y}{\sqrt{1 - \frac{u^2}{c^2}}} \\ u_3 &= \frac{dx_3}{d\tau} = \frac{dx_3}{dt} \cdot \frac{dt}{d\tau} = \frac{u_z}{\sqrt{1 - \frac{u^2}{c^2}}} \end{aligned} \right\} \text{using (5.44)} \quad \dots(5.45a)$$

and

$$u_4 = \frac{dx_4}{d\tau} = \frac{d(ict)}{d\tau} = ic \frac{dt}{d\tau} = \frac{ic}{\sqrt{1 - \frac{u^2}{c^2}}}$$

or

$$\text{in brief } \underline{u} = \left(\frac{\underline{u}}{\sqrt{1 - \frac{u^2}{c^2}}}, \frac{ic}{\sqrt{1 - \frac{u^2}{c^2}}} \right) \quad \dots(5.45b)$$

where $\underline{u} = \frac{d\mathbf{r}}{dt}$ is the usual three dimensional velocity vector.

The four velocity is a four vector which lies in the direction of the tangent of the time-track and the norm of this vector is given by

$$\underline{u} \cdot \underline{u} = \frac{u^2}{1 - \frac{u^2}{c^2}} - \frac{c^2}{1 - \frac{u^2}{c^2}} = -c^2 \quad \dots(5.46)$$

Thus the four-velocity is a time like vector of constant magnitude and differentiating eqn. (5.46) we get

$$\underline{u} \cdot \frac{d\underline{u}}{d\tau} = 0. \quad \dots(5.47)$$

Acceleration four vector : The acceleration four-vector is defined as

$$\underline{a} = \frac{d\underline{u}}{d\tau} = \frac{d\underline{u}}{dt} \left(1 - \frac{u^2}{c^2} \right)^{-1/2} \quad \dots(5.48)$$

In brief the acceleration four vector of four acceleration is written as

$$\underline{a} = \left(\frac{c^2}{c^2 - u^2} \underline{a} + \underline{u} \frac{c^2 (\underline{u} \cdot \underline{a})}{(c^2 - u^2)^2}; \frac{ic^3 (\underline{u} \cdot \underline{a})}{(c^2 - u^2)^2} \right) \quad \dots(5.49)$$

where $\underline{a} = \frac{d\underline{u}}{dt}$ is the usual three dimensional acceleration of the particle. According to eqn. (5.47) the four vectors $\underline{u}$ and $\underline{a}$ are perpendicular to each other.

Energy-Momentum four vector : The momentum four vector or four momentum is obtained by multiplying the velocity four-vector by the rest mass m_0 so that

$$\left. \begin{aligned}
 p_1 &= m_0 u_1 = \frac{m_0 u_x}{\sqrt{1 - \frac{u^2}{c^2}}} \\
 p_2 &= m_0 u_2 = \frac{m_0 u_y}{\sqrt{1 - \frac{u^2}{c^2}}} \\
 p_3 &= m_0 u_3 = \frac{m_0 u_z}{\sqrt{1 - \frac{u^2}{c^2}}} \\
 p_4 &= m_0 u_4 = \frac{im_0 c}{\sqrt{1 - \frac{u^2}{c^2}}} = imc = \frac{iE}{c}
 \end{aligned} \right\} \dots(5.45a)$$

and

In brief the momentum four vector is written as

$$\mathbf{p} = \left(\mathbf{p}, \frac{iE}{c} \right) \dots(5.50b)$$

where

$$\mathbf{p} = \frac{m_0 \mathbf{u}}{\sqrt{1 - \frac{u^2}{c^2}}}$$

is the usual three dimensional momentum vector.

The norm of the momentum four-vector is given by

$$\mathbf{p} \cdot \mathbf{p} = p^2 - \frac{E^2}{c^2} = -m_0^2 c^2 \dots(5.51)$$

Thus the four-momentum is like a time like vector.

Aliter :

Energy-Momentum four vector : The momentum $\mathbf{p}$ is defined as

$$\mathbf{p} = m\mathbf{u} \text{ (classically)} \dots(i)$$

$$= \frac{m_0 \mathbf{u}}{\sqrt{1 - \frac{u^2}{c^2}}} \text{ (relativistically)} \dots(ii)$$

We know that

$$ds^2 = -(dx^2 + dy^2 + dz^2) + c^2 dt^2$$

This implies

$$\left(\frac{dx}{dt} \right)^2 = - \left[\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 + \left(\frac{dz}{dt} \right)^2 \right] + c^2 = c^2 - u^2$$

or

$$\frac{ds}{dt} = \sqrt{c^2 - u^2} = c \sqrt{1 - \frac{u^2}{c^2}}$$

or

$$\frac{ds}{c} = dt \sqrt{1 - \frac{u^2}{c^2}} = \frac{dt}{\gamma} \text{ where } \gamma = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}$$

$$\gamma ds = c dt$$

...(iii)

The cartesian equivalent to equation (ii) is

$$p_x = \frac{m_0 u_x}{\sqrt{1 - \frac{u^2}{c^2}}} = \gamma m_0 u_x = \gamma m_0 \frac{dx}{dt} = m_0 \frac{c}{ds} \frac{dx}{ds} \text{ using (iii)}$$

or $p_x = m_0 c \frac{dx}{ds}$

Similarly

$$p_y = m_0 c \frac{dy}{ds} \quad \text{and} \quad p_z = m_0 c \frac{dz}{ds}$$

Also

$$E = mc^2 = \frac{m_0 c^2}{\sqrt{1 - \frac{u^2}{c^2}}} = \gamma m_0 c^2 = m_0 c^2 \cdot \frac{c}{ds} \frac{dt}{ds}$$

or

$$\frac{iE}{c} = m_0 c \frac{ic dt}{ds}$$

Writing (p_1, p_2, p_3, p_4) for $(p_x, p_y, p_z, \frac{iE}{c})$ and (x^1, x^2, x^3, x^4) for (x, y, z, ict) respectively, we get

$$p_1 = m_0 c \frac{dx^1}{ds}, p_2 = m_0 c \frac{dx^2}{ds}, p_3 = m_0 c \frac{dx^3}{ds}, p_4 = m_0 c \frac{dx^4}{ds} \quad \dots(iv)$$

From this it follows that the set of four quantities $(p_1, p_2, p_3, \frac{iE}{c})$ are the components of energy-momentum four-vector and expressed as $\mathbf{p} = (p_x, p_y, p_z, \frac{iE}{c}) = (\mathbf{p}, \frac{iE}{c}) = p^\mu$

Force four vector or four force : The force is defined as the rate of change of momentum. So the force four vector is given by

$$\underline{\mathbf{F}} = \frac{d\mathbf{p}}{d\tau} = m_0 \frac{d\mathbf{u}}{(d\tau)} \quad \dots(5.52)$$

This is the equation of motion in Minkowski space.

The components of the four-force are written as

$$\left. \begin{aligned} F_1 &= \frac{dp_1}{d\tau} = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} \frac{dp_1}{dt} \\ F_2 &= \frac{dp_2}{d\tau} = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} \frac{dp_2}{dt} \\ F_3 &= \frac{dp_3}{d\tau} = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} \frac{dp_3}{dt} \\ F_4 &= \frac{dp_4}{d\tau} = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} \frac{dp_4}{dt} \end{aligned} \right\} \quad \dots(5.53a)$$

Substituting values of p_1, p_2, p_3 and p_4 from (50a), we may write the four-force as

$$\underline{F} = \left\{ \frac{\mathbf{F}}{\sqrt{1 - \frac{u^2}{c^2}}}, \frac{i(\mathbf{F} \cdot \mathbf{u})/c}{\sqrt{1 - \frac{u^2}{c^2}}} \right\} \quad \dots(5.53b)$$

where $\mathbf{F} = \frac{d\mathbf{p}}{dt}$ is the usual three dimensional force vector.

16.33. RELATIVISTIC FORMULATION OF EQUATION OF MOTION : MINKOWSKI EQUATION OF MOTION :

In Newtonian Mechanics, the equation of motion of a particle of linear momentum $\mathbf{p}$ and rest mass m_0 is given by

$$\underline{F} = \frac{d\mathbf{p}}{dt} = \mathbf{p} = \frac{d}{dt}(m_0 \mathbf{v}) \quad \dots(1)$$

As time t of particle vectors from one inertial frame to another. Therefore this expression is not Lorentz invariant. Here we try to generalise equation (1) so that it may satisfy the principle of velocity. In Newtonian Mechanics time t plays the role of invariant parameter; but in relativity time t is variable and similar parameter is length ds , which is given by

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

This implies

$$\left(\frac{dx}{dt}\right)^2 = c^2 - \left\{ \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 \right\}$$

$$= c^2 - u^2$$

where u is ordinary velocity of particle $u^2 = \sqrt{u_x^2 + u_y^2 + u_z^2}$

$$\Rightarrow \frac{ds}{dt} = \sqrt{c^2 - u^2} = c \sqrt{1 - \frac{u^2}{c^2}} \quad \dots(2a)$$

$$\Rightarrow \frac{ds}{c} = dt \sqrt{1 - \frac{u^2}{c^2}} \quad \text{or} \quad \frac{ds}{c} = \frac{dt}{\gamma} \quad \dots(2b)$$

$$\Rightarrow ds = \frac{c dt}{\gamma} \quad \dots(3)$$

Physical significance of ds is that apart from the factor c , it is proper time of the particle in a frame in which it is instantaneously at rest.

The momentum components in cartesian coordinates may be expressed as

$$p_x = m u_x = \frac{m_0 u_x}{\sqrt{1 - \frac{u^2}{c^2}}} = \gamma m_0 u_x$$

$$= \gamma m_0 \frac{dx}{dt} = m_0 c \frac{dx}{ds} \quad (\text{since } c dt = \gamma ds)$$

Similarly

$$p_y = m_0 c \frac{dy}{ds}$$

$$p_z = m_0 c \frac{dz}{ds}$$

In four dimensional space, the result may be generalised as

four momentum

$$p^\mu = m_0 c \frac{dx^\mu}{ds} \quad \dots(4)$$

Therefore in analogy with Newtonian mechanics, the equation of motion in relativity would become

$$p^{\cdot\mu} = K^{\mu} \quad \dots(5)$$

where K^{μ} is four force of Minkowski force and dot means differentiation with respect to invariant parameter 's'.

$$\Rightarrow \frac{d}{ds} \left(m_0 c \frac{dx^{\mu}}{ds} \right) = K^{\mu} \quad \dots(6)$$

This equation is called *relativistic equation of motion* or *Minkowski equation of motion*.

Classical limit of Minkowski equation of motion :

In classical limit $\frac{v}{c} \ll 1$.

As for any function f ,

$$\frac{df}{ds} = \frac{df}{dt} \cdot \frac{dt}{ds}$$

so equation (6) maybe expressed as

$$\frac{d}{dt} \left(m_0 c \frac{dx^{\mu}}{ds} \right) \cdot \frac{dt}{ds} = K^{\mu}$$

or

$$\frac{d}{dt} \left(m_0 c \frac{dx^{\mu}}{ds} \right) = K^{\mu} \frac{ds}{dt} \quad \dots(7)$$

Again writing $\frac{dx^{\mu}}{ds} = \frac{dx^{\mu}}{dt} \cdot \frac{dt}{ds}$, equation (7) takes the form

$$\frac{d}{dt} \left(m_0 c \frac{dx^{\mu}}{dt} \cdot \frac{dt}{ds} \right) = K^{\mu} \frac{ds}{dt} \quad \dots(8)$$

The space part of equation (8) may be expressed as

$$\frac{d}{dt} \left(m_0 c \frac{dt}{ds} \cdot \frac{dx_i}{dt} \right) = K_i \frac{ds}{dt}, \quad \text{where } i = 1, 2, 3 \quad \dots(9)$$

Comparing this with Newtonian equation of motion :

$$\frac{d}{dt} \left(m \frac{dx_i}{dt} \right) = F_i \quad \dots(10)$$

we note that $m = m_0 c \frac{dt}{ds} = \frac{m_0}{\sqrt{1 - \frac{u^2}{c^2}}}$ using (2a) $\dots(11)$

and

$$K_i \frac{ds}{dt} = F_i$$

$$\text{i.e., } K \frac{ds}{dt} = F \quad \dots(12)$$

where K is spatial part of four vector K^{μ} . Further in the limit

$$\frac{v}{c} \rightarrow 0, \frac{ds}{dt} = c$$

and

$$K = \frac{F}{c} \quad \dots(13)$$

i.e., K and F are essentially the same. Equation (11) implies that mass varies with velocity of particle; but there is no internal mechanism to increase the mass of the particle, however it is due to equivalence of mass and energy. Accordingly the increase in mass is due to increased

kinetic energy of particle. This suggests that we can correct the classical equation of motion by using a variable mass m in place of rest mass m_0 . But the Minkowski form of equation is the fundamental relativistic formulation of equation of motion.

16.34. RELATIVISTIC CLASSIFICATION OF PARTICLES :

According to energy-momentum four vector

$$\left(p_1, p_2, p_3, \frac{iE}{c}\right)$$

$$E^2 - p^2 c^2 = m_0^2 c^4. \quad \dots(1)$$

Let us now represent the energy-momentum four vector in Minkowski space, the two dimensional section is shown in fig. 5.10 for convenience. The energy momentum trace out a

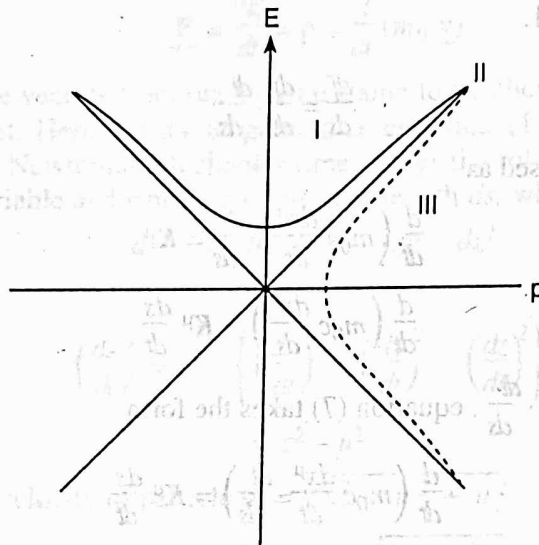


Fig. 5.10

hyperbola. There are the three possibilities about the classification of particles according as

$$E^2 - p^2 c^2 > 0, = 0 \quad \text{or} \quad < 0$$

1. The particles for which $E^2 - p^2 c^2 > 0$:

When $E^2 - p^2 c^2 > 0$; then according to eq. (1); $m_0^2 > 0$. Obviously the rest mass m_0 of this class of particles is real and positive. In energy momentum diagram such particles are represented by hyperbolas in the forward light cone (region I). This class of particles consists of all those particles which travel with a velocity u always less than c . The material particles belong to this class.

2. The particles for which $E^2 - p^2 c^2 = 0$:

When $E^2 - p^2 c^2 = 0$; then according to eqn. (1), $m_0^2 = 0$. i.e., this class consists of the particles of **zero rest mass**. These particles travel with velocity of light and are represented, in energy momentum diagram, by the degenerate limit of the hyperbolas in the upper half of the light cone (Region II). The photons and the four kinds of neutrinos (identified in the weak interactions) belong to this class. The momentum of the particles of this class is no longer concerned with the velocities; but with the energy, given by

$$p = \frac{E}{c}$$

3. The particles for which $E^2 - p^2 c^2 < 0$:

When $E^2 - p^2 c^2 < 0$; then according to equation (1), m_0^2 is negative. consequently the rest mass of this class of particles is imaginary. These particles always travel with a velocity u

greater than c . The particles of this class are known as **tachyons**. The existence of the tachyons (faster than light particles) is not contrary to the special theory of relativity.

Ex. 37. Show that the four dimensional volume element " $dx dy dz dt$ " is invariant under Lorentz transformations.

Solution :

According to length contraction, we have

$$dx' = dx \sqrt{1 - \frac{v^2}{c^2}}$$

$$dy' = dy$$

$$dz' = dz$$

According to time dilation, $dt' = \frac{dt}{\sqrt{1 - \frac{v^2}{c^2}}}$

$\therefore$ Four - dimensional volume element in system S'

$$= dx dy' dz' dt'$$

$$= \left\{ dx \cdot \sqrt{1 - \frac{v^2}{c^2}} \right\} \cdot dy \cdot dz \left\{ \frac{dt}{\sqrt{1 - \frac{v^2}{c^2}}} \right\}$$

$$= dx dy dz dt$$

$$= \text{four dimensional volume element in system } S.$$

16.35. Covariant Four-Dimensional Formulation of the Laws of Mechanics

If the form of a law is not changed by certain coordinate transformation, the law is said to be *invariant* or *covariant*. If any physical law may be expressed in a covariant *four dimensional form*, then the law will be invariant under Lorentz transformations.

In four dimensions the position vector will be termed as position four vector. The position four vector in four dimensions will be represented as

$$\underline{r} = \hat{x}_1 x_1 + \hat{x}_2 x_2 + \hat{x}_3 x_3 + \hat{x}_4 x_4$$

where $\hat{x}_1, \hat{x}_2, \hat{x}_3$ and $\hat{x}_4$ are unit vectors along x_1, x_2, x_3 and x_4 respectively. To differentiate position four vector with ordinary position vector we have to put $\underline{r}$ in place of $\underline{r}$.

Taking dot product of $\underline{r}$ with itself, we shall get a world scalar and hence Lorentz invariant, i.e.,

$$\begin{aligned} \underline{r} \cdot \underline{r} &= (\hat{x}_1 x_1 + \hat{x}_2 x_2 + \hat{x}_3 x_3 + \hat{x}_4 x_4) \cdot (\hat{x}_1 x_1 + \hat{x}_2 x_2 + \hat{x}_3 x_3 + \hat{x}_4 x_4) \\ &= x_1^2 + x_2^2 + x_3^2 + x_4^2 = \text{Lorentz invariant.} \end{aligned}$$

If $d\underline{r}$ represents the change in position four vector, then we have

$$d\underline{r} = \hat{x}_1 dx_1 + \hat{x}_2 dx_2 + \hat{x}_3 dx_3 + \hat{x}_4 dx_4$$

taking dot product of this with itself, we shall get a world scalar and hence Lorentz invariant, i.e.,

$$d\underline{r} \cdot d\underline{r} = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 = \text{Lorentz invariant.}$$

or $d\underline{r}^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2$

But $x_1 = x, x_2 = y, x_3 = z$ and $x_4 = ict$

$$\therefore d\underline{r} \cdot d\underline{r} = d\underline{r}^2 = dx^2 + dy^2 + dz^2 - c^2 dt^2 = \text{Lorentz invariant.} \quad \dots(1)$$

Let us consider a system in which a particle is momentarily at rest. As the object is at rest, its displacement vanishes, i.e., $dx = 0, dy = 0, dz = 0$ and let the time be denoted by τ , then we have

$$d\mathbf{r}^2 = -c^2 d\tau^2 \quad \dots(2)$$

As $d\mathbf{r}^2$ is an invariant, we must have

$$d\mathbf{r}^2 = dx^2 + dy^2 + dz^2 - c^2 dt^2 = -c^2 d\tau^2 \quad \dots(3)$$

From equation (2) it is clear that $d\tau = -d\mathbf{r}/ic$ is an invariant $d\mathbf{r}$ and c are invariant.

The time $d\tau$, as measured in the rest frame is called the *proper time*.

From equation (3) we have

$$d\tau^2 = -\frac{1}{c^2} (dx^2 + dy^2 + dz^2 - c^2 dt^2)$$

$$= \frac{dt^2}{c^2} \left[c^2 - \left\{ \left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 + \left(\frac{dz}{dt} \right)^2 \right\} \right]$$

$$\text{or} \quad d\tau = dt \sqrt{\left(1 - \frac{v^2}{c^2} \right)}$$

$$\text{or} \quad d\tau = \frac{dt}{\sqrt{1 - v^2/c^2}}$$

which is an expression for *time dilation*.

As one of the components of a four vector is imaginary, the square of four vector may be positive and negative in like ordinary vector. If the square of magnitude of a vector is greater than or equal to zero, the four-vector is said to be *space like*; but if the magnitude is negative the four vector is said to be *time like*.

As the magnitudes of the four-vectors are world scalars, the designations are unaffected by Lorentz transformations.

Let us now consider a difference vector between two world points which may be either +ve or -ve. Let $\mathbf{R}$ be the difference vector defined as

$$\mathbf{R} = \mathbf{r}_2 - \mathbf{r}_1,$$

where $\mathbf{r}_1$, and $\mathbf{r}_2$ are the position four-vectors of the world points, respectively.

Then the magnitude $\mathbf{R}$ is given by

$$\mathbf{R}^2 = |\mathbf{r}_2 - \mathbf{r}_1|^2 = |\mathbf{r}_2 - \mathbf{r}_1|^2 - c^2 (t_2 - t_1)^2,$$

where $\mathbf{r}_1$ and $\mathbf{r}_2$ are the ordinary position vectors.

Thus $\mathbf{R}$ will be space like if

$$|(\mathbf{r}_2 - \mathbf{r}_1)^2| \geq c^2 (t_2 - t_1)^2 \quad \dots(4)$$

and $\mathbf{R}$ will be time like if

$$|(\mathbf{r}_2 - \mathbf{r}_1)|^2 < c^2 (t_2 - t_1)^2 \quad \dots(5)$$

Let the spatial axes be so oriented that the difference vector $(\mathbf{r}_2 - \mathbf{r}_1)$ is along X-axes and therefore equal to $x_2 - x_1$. Then the Lorentz transformation of fourth component of $\mathbf{R}$ may be written as

$$c (t_2' - t_1') = \frac{c (t_2 - t_1) - (v/c) (x_2 - x_1)}{\sqrt{1 - v^2/c^2}}$$

like
$$t' = \frac{t - \frac{vx}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

If $\underline{R}$ is space like, then

$$c(t_2 - t_1) < x_2 - x_1$$

and hence it is possible to find a velocity $v < c$ such that $ic(t_2' - t_1')$ vanishes. The vanishing of $ic(t_2' - t_1')$ i.e., fourth component of $\underline{R}$, means that if the distance between two events is space-like, it is possible to find a system in which the two events are simultaneous.

Now in order to illustrate the process of restating a physical law in a covariant four dimensional formulation, let us take the scalar wave equation of the type

$$\nabla^2 \psi - \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} = 0 \quad \dots(6)$$

Like three dimensional gradient, we may consider a four dimensional differential operator in world space, denoted by symbol $\square$ and known as **D' Alembertian** with components

$$\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}$$

i.e.,
$$\square = \hat{i} \frac{\partial}{\partial x_1} + \hat{j} \frac{\partial}{\partial x_2} + \hat{k} \frac{\partial}{\partial x_3} + \hat{x} \frac{\partial}{\partial x_4} \quad \dots(7)$$

Taking scalar product $\square$ with itself we get a scalar quantity and hence Lorentz invariant, i.e.,

$$\square \cdot \square = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} + \frac{\partial^2}{\partial x_4^2} \right) = \text{Lorentz invariant.}$$

But $x_1 = x, x_2 = y, x_3 = z, x_4 = ict$

$$\therefore \square^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) = \text{Lorentz invariant.}$$

Thus equation (6) is invariant under Lorentz transformations.

Clearly, *any physical law may be invariant under Lorentz transformation, if the law may be expressed as a covariant four dimensional form.*

16.36. Lorentz Transformation of Space and Time in Four-vector Form

Consider two systems S and S' , the latter moving with velocity v relative to former along positive direction of X -axis. In Minkowski space let the co-ordinates of an event be represented by the quantities (x, y, z, ict) or x_μ ($\mu = 1, 2, 3, 4$) where

$$\left. \begin{aligned} x_1 &= x \\ x_2 &= y \\ x_3 &= z \\ x_4 &= ict \end{aligned} \right\} \quad \dots(1)$$

Lorentz transformation of space and time with $\beta = v/c$ are written as

$$x' = \frac{x - vt}{\sqrt{1 - \beta^2}}$$

$$y' = y$$

$$z' = z$$

and

$$t' = \frac{t - \frac{vx}{c^2}}{\sqrt{1 - \beta^2}}$$

or equivalently

$$x_1' = \frac{x_1 - vt}{\sqrt{1 - \beta^2}} = \gamma(x_1 - i\beta x_4)$$

$$x_2' = x_2$$

$$x_3' = x_3$$

$$\text{with } \gamma = \frac{1}{\sqrt{1 - \beta^2}} \quad \dots(2)$$

and

$$x_4' = \frac{\left(t - \frac{vx_1}{c^2}\right)}{\sqrt{1 - \beta^2}} = \gamma(x_4 - i\beta x_1)$$

Above transformations can be expressed as

$$x_1' = \gamma x_1 + 0 \cdot x_2 + 0 \cdot x_3 + i\beta \gamma x_4$$

$$x_2' = 0 \cdot x_1 + 1 \cdot x_2 + 0 \cdot x_3 + 0 \cdot x_4$$

$$x_3' = 0 \cdot x_1 + 0 \cdot x_2 + 1 \cdot x_3 + 0 \cdot x_4$$

$$x_4' = -i\beta \gamma x_1 + 0 \cdot x_2 + 0 \cdot x_3 + \gamma x_4 \quad \dots(3)$$

These results can be written compactly by a single equation

$$x_\mu' = \alpha_{\mu\nu} x_\nu$$

where

$v \rightarrow$	1	2	3	4
$\mu \downarrow$	1	2	3	4
$\alpha_{\mu\nu} =$	1	2	3	4
	γ	0	0	$i\beta\gamma$
	2	1	0	0
	3	0	1	0
	4	$-i\beta\gamma$	0	γ

Equation (4) represents the Lorentz transformations of space and time in four-vector form.

Similarly Lorentz transformations of any four-vector A_μ ($\mu = 1, 3, 4$) maybe expressed as

$$A_\mu' = a_{\mu\nu} A_\nu$$

16-37. Orthogonality of Lorentz transformation in Minkowski SpaceThe Lorentz transformation equations in Minkowski space $x_1 = x$, $x_2 = y$, $x_3 = z$, $x_4 = i ct$ are given by

$$x_1' = \gamma \left[x_1 + \frac{i v x_4}{c} \right], \quad x_2' = x_2, \quad x_3' = x_3 \quad \text{and} \quad x_4' = \gamma \left[x_4 - \frac{i v x_1}{c} \right]$$

$$\text{where } \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

These transformations may be expressed as

$$x'_\mu = \alpha_{\mu\nu} x_\nu, \mu = 1, 2, 3, 4$$

with $\alpha_{\mu\nu}$ = real for $\mu, \nu = 1, 2, 3$ and α_{44} is real while $\alpha_{\mu 4}$ and $\alpha_{4\mu}$, imaginary.

$$\alpha_{\mu\nu} = \begin{matrix} \nu \rightarrow & 1 & 2 & 3 & 4 \\ \mu \downarrow & & & & \\ 1 & \gamma & 0 & 0 & i\beta\gamma \\ 2 & 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 1 & 0 \\ 4 & -i\beta\gamma & 0 & 0 & 1 \end{matrix}$$

The invariant quantity $s^2 = x_\mu x_\mu = x'_\mu x'_\mu$
 $= \alpha_{\mu\nu} \alpha_{\mu\sigma} x_\nu x_\sigma$

Clearly $\alpha_{\mu\nu} \alpha_{\mu\sigma} = \delta_{\nu\sigma}$
 and $\alpha_{\nu\mu} \alpha_{\sigma\mu} = \delta_{\sigma\nu}$

These conditions are termed as orthogonality conditions satisfied by coefficient of transformation matrix $\alpha_{\mu\nu}$ for s^2 to be invariant. It shows that the Lorentz transformations in Minkowski space are orthogonal.

16-38. Lorentz Transformations of Electromagnetic Potentials A and ϕ

The Lorentz condition is

$$\text{div } \mathbf{A} + \frac{1}{c^2} \frac{\partial \phi}{\partial t} = 0 \quad \dots(1)$$

This equation suggests that we can define a four vector A_μ as

$$A_\mu = \left(\mathbf{A}, \frac{i\phi}{c} \right) \quad \dots(2)$$

To prove that A_μ is a four vector, let us consider Maxwell's field equations in terms of electromagnetic potentials A and ϕ

$$\square^2 \mathbf{A} = \mu_0 \mathbf{j} \quad \dots(3)$$

and $\square^2 \phi = \frac{\rho}{\epsilon_0} \quad \dots(4)$

These equations can be written as

$$\square^2 A_1 = -\mu_0 j_1 \quad \dots(5a)$$

$$\square^2 A_2 = -\mu_0 j_2 \quad \dots(5b)$$

$$\square^2 A_3 = -\mu_0 j_3 \quad \dots(5c)$$

$$\square^2 \phi = -\frac{\rho}{\epsilon_0} \quad \dots(5d)$$

The last equation can be written as

$$\square^2 A_4 = \frac{i\rho}{c\epsilon_0} = -\frac{ic\rho}{c^2\epsilon_0} = -\mu_0 (ic\rho) \left(\text{since } \mu_0\epsilon_0 = \frac{1}{c^2} \right)$$

i.e.,

$$\square^2 A_4 = -\mu_0 j_4 \quad (\text{since } j_4 = ic\rho)$$

Thus complete set of equations (5) is written as

$$\square^2 A_\mu = -\mu_0 j_\mu \quad (\mu = 1, 2, 3, 4) \quad \dots(6)$$

Now as $\square^2$ is Lorentz invariant and j_μ is a four factor, A_μ must be a four vector. This four vector is known as *four vector potential* or *electromagnetic four potential*. This must transform according to

$$A'_\mu = \alpha_{\mu\nu} A_\nu \quad \dots(7a)$$

Where $\alpha_{\mu\nu} =$

$$\begin{bmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{bmatrix}$$

So that

$$A'_1 = \alpha_{1\nu} A_\nu$$

$$= \alpha_{11} A_1 + \alpha_{12} A_2 + \alpha_{13} A_3 + \alpha_{14} A_4$$

$$= \gamma A_1 + 0.A_2 + 0.A_3 + i\beta\gamma.A_4$$

$$= \gamma (A_1 + i\beta A_4)$$

$$= \frac{A_1 + i \frac{v}{c} \left(\frac{i\phi}{c} \right)}{\sqrt{(1 - \beta^2)}} \quad \dots(8)$$

or

$$A'_1 = \frac{A_1 - \frac{v}{c^2} \phi}{\sqrt{(1 - \beta^2)}} \quad \dots(8a)$$

$$A'_2 = \alpha_{2\nu} A_\nu = \alpha_{21} A_1 + \alpha_{22} A_2 + \alpha_{23} A_3 + \alpha_{24} A_4$$

$$= 0.A_1 + 1.A_2 + 0.A_3 + 0.A_4$$

or

$$A'_2 = A_2 \quad \dots(8b)$$

$$A'_3 = \alpha_{3\nu} A_\nu$$

$$= \alpha_{31} A_1 + \alpha_{32} A_2 + \alpha_{33} A_3 + \alpha_{34} A_4$$

$$= 0.A_1 + 0.A_2 + 1.A_3 + 0.A_4$$

$$A'_3 = A_3 \quad \dots(8c)$$

and

$$A'_4 = \alpha_{4\nu} A_\nu = (\alpha_{41} A_1 + \alpha_{42} A_2 + \alpha_{43} A_3 + \alpha_{44} A_4)$$

$$= -i\beta\gamma A_1 + 0.A_2 + 0.A_3 + \gamma A_4$$

$$= \gamma A_4 - i\beta\gamma A_1 \quad \dots(8d i)$$

or

$$\frac{i\phi}{c} = \frac{\frac{i\phi}{c} - i \frac{v}{c} A_1}{\sqrt{(1 - \beta^2)}}$$

or

$$\phi' = \frac{\phi - v A_1}{\sqrt{(1 - \beta^2)}} \quad \dots(8d ii)$$

Thus the transformation laws for electromagnetic potentials A and ϕ are written as

$$\left. \begin{aligned} A'_1 &= \frac{A_1 - \frac{v}{c^2} \phi}{\sqrt{(1 - \beta^2)}}, A'_2 = A_2, A'_3 = A_3 \\ \phi' &= \frac{\phi - v A_1}{\sqrt{(1 - \beta^2)}} \end{aligned} \right\} \quad \dots(8)$$

and

The inverse transformation for electromagnetic potentials A and ϕ are obtained by replacing v by $-v$ and interchanging primed and unprimed quantities, i.e.,

$$\left. \begin{aligned} A_1' &= \frac{A_1 + \frac{v}{c} \phi'}{\sqrt{(1 - \beta^2)}}, \quad A_2 = A_2', \quad A_3 = A_3' \\ \phi &= \frac{\phi' + v A_1'}{\sqrt{(1 - \beta^2)}} \end{aligned} \right\} \dots(9)$$

and

Equations (8) and (9) represent required transformations for four vector potential.

16.39. Lorentz Condition in Covariant form

The Lorentz condition is

$$\text{div } A + \frac{1}{c^2} \frac{\partial \phi}{\partial t} = 0$$

This can be written as

$$\text{div } A + \frac{\partial(i\phi/c)}{\partial(ict)} = 0$$

$$\text{i.e.,} \quad \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3} + \frac{\partial A_4}{\partial x_4} = 0$$

$$\text{i.e.,} \quad \frac{\partial A_\mu}{\partial x_\mu} = \square \cdot A_\mu = 0 \quad \dots(1)$$

$$\text{where} \quad \square \cdot = \frac{\partial}{\partial x_\mu} \text{ is four dimensional divergence operator.}$$

Equation (1) is covariant equation and is known as *Lorentz condition in covariant form*. This equation expresses that the four-divergence of the electromagnetic four-potential vanishes.

16.40. Covariance of Maxwell Field Equations under Lorentz Transformations

By invariance (or covariance) of Maxwell field equations we mean that these equations do not change their form by the change of the frame of reference. Here we shall prove the invariance of Maxwell equations under Lorentz transformations in terms of four vectors.

Maxwell field equations are

$$\text{div } D = \rho \quad \dots(a)$$

$$\text{div } B = 0 \quad \dots(b)$$

$$\text{curl } E = -\frac{\partial B}{\partial t} \quad \dots(c)$$

$$\text{curl } H = j + \frac{\partial D}{\partial t} \quad \dots(d)$$

$$\text{with } D = \epsilon_0 E$$

$$\text{and } B = \mu_0 H$$

In terms of electromagnetic potentials A and ϕ , these equations are written as

$$\left. \begin{aligned} \square^2 A &= -\mu_0 j \\ \square^2 \phi &= -\frac{\rho}{\epsilon_0} \end{aligned} \right\} \dots(3)$$

with the Lorentz condition

$$\text{div } A + \frac{1}{c^2} \frac{\partial \phi}{\partial t} = 0 \quad \dots(4)$$

The electromagnetic four potential A_μ and the current four potential j_μ are defined as

$$A_\mu = \left(A, \frac{i\phi}{c} \right) \quad \dots(5)$$

$$j_\mu = (j, icp). \quad \dots(6)$$

These four vectors obey the Lorentz transformations

$$A'_\mu = \alpha_{\mu\nu} A_\nu \quad \dots(8)$$

and

$$j'_\mu = \alpha_{\mu\nu} j_\nu$$

$$\text{where } \alpha_{\mu\nu} = \begin{bmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{bmatrix}$$

i.e., the components of electromagnetic four potential and current four potential transform according to the relations

$$\left. \begin{aligned} A'_1 &= \frac{A_1 - \frac{v}{c^2} \phi}{\sqrt{1 - \beta^2}}, \quad A'_2 = A_2, \quad A'_3 = A_3 \\ A'_4 &= \frac{A_4 - i\beta A_1}{\sqrt{1 - \beta^2}} \quad \text{or } \phi' = \frac{\phi - v A_1}{\sqrt{1 - \beta^2}} \end{aligned} \right\} \quad \dots(10)$$

and

$$\left. \begin{aligned} j'_1 &= \frac{j_1 - vp}{\sqrt{1 - \beta^2}}, \quad j'_2 = j_2, \quad j'_3 = j_3 \\ j'_4 &= \frac{j_4 - i\beta j_1}{\sqrt{1 - \beta^2}} \quad \text{or } \rho' = \frac{\rho - \frac{v}{c^2} j_1}{\sqrt{1 - \beta^2}} \end{aligned} \right\} \quad \dots(11)$$

In terms of electromagnetic four potential A_μ and current four potential j_μ Maxwell's equations take the form

$$\square^2 A_\mu = -\mu_0 j_\mu \quad \dots(12)$$

With the Lorentz condition

$$\frac{\partial A_\mu}{\partial x_\mu} = 0 \quad \dots(13)$$

The invariance of Maxwell's equations requires that in any inertial frame S' equations (12) and (13) must retain the same form, i.e., Maxwell's field equations will be invariant if

$$\left. \begin{aligned} \square'^2 A'_\mu &= -\mu_0 j'_\mu \\ \text{With the condition} \\ \frac{\partial A'_\mu}{\partial x'_\mu} &= 0. \end{aligned} \right\} \quad \dots(14)$$

Let us now consider

$$(a) \quad \square'^2 A'_1 = \square^2 A'_1 \quad [\text{since } \square'^2 = \square^2]$$

$$= \square^2 \left[\frac{A_1 - \frac{v}{c^2} \phi}{\sqrt{(1 - \beta^2)}} \right] \quad \text{(from 10)}$$

$$= \frac{\square^2 A_1 - \frac{v}{c^2} \square^2 \phi}{\sqrt{(1 - \beta^2)}} = \frac{\left[-\mu_0 j_1 - \frac{v}{\epsilon_0} \left(-\frac{\rho}{\epsilon_0} \right) \right]}{\sqrt{(1 - \beta^2)}} = \frac{\mu_0 (j_1 - v\rho)}{\sqrt{(1 - \beta^2)}} \quad \text{(since } \mu_0 \epsilon_0 = 1/c^2)$$

$$= -\mu_0 j_1' \quad \text{(from 11)}$$

$$\text{i.e.,} \quad \square'^2 A_1' = -\mu_0 j_1' \quad \dots(15a)$$

$$(b) \quad \square'^2 A_2' = \square^2 A_2' \quad \text{(Since } \square'^2 = \square^2)$$

$$= \square^2 A_2 \quad \text{(Since } A_2' = A_2 \text{ from 10)}$$

$$= -\mu_0 j_2 \quad \text{(from 12)}$$

$$\text{i.e.,} \quad \square'^2 A_2' = -\mu_0 j_2' \quad \text{(since } j_2' = j_2 \text{ from 11)} \quad \dots(15b)$$

$$(c) \quad \square'^2 A_3' = -\square^2 A_3 \quad \text{(since } \square'^2 = \square^2 \text{ and } A_3' = A_3)$$

$$= -\mu_0 j_3 \quad \text{(from 12)}$$

$$= -\mu_0 j_3' \quad \text{[(since } j_3' = j_3 \text{ from (11))} \dots(15c)$$

$$(d) \quad \square'^2 A_4' = \square^2 A_4' \quad \text{(since } \square'^2 = \square^2)$$

$$= \square^2 \left[\frac{A_4 - i\beta A_1}{\sqrt{(1 - \beta^2)}} \right] \quad \text{(from 10)}$$

$$= \frac{\square^2 A_4 - i\beta \square^2 A_1}{\sqrt{(1 - \beta^2)}} = \frac{-\mu_0 j_4 - i\beta (-\mu_0 j_1)}{\sqrt{(1 - \beta^2)}}$$

$$= -\frac{\mu_0 (j_4 - i\beta j_1)}{\sqrt{(1 - \beta^2)}} = -\mu_0 j_4' \quad \text{from (11)}$$

$$\text{i.e.,} \quad \square'^2 A_4' = -\mu_0 j_4'$$

Equations (15a), (15b), (15c) and (15d) can be written in the form of a single equation as

$$\square' A_\mu' = -\mu_0 j_\mu'$$

Further we have

$$A_\mu' = \alpha_{\mu\nu} A_\nu$$

$$A_\mu' = \frac{\partial x_\mu'}{\partial x_\nu} A_\nu$$

or

$$\frac{\partial A_\mu'}{\partial x_\mu'} = \frac{\partial}{\partial x_\mu'} \frac{\partial x_\mu'}{\partial x_\nu} A_\nu$$

i.e.,

$$\frac{\partial A_\mu'}{\partial x_\mu'} = \frac{\partial A_\nu}{\partial x_\nu} \quad \dots(17)$$

But from Lorentz condition (13) $\frac{\partial A_\nu}{\partial x_\nu} = 0$.

$$\therefore \frac{\partial A_\mu'}{\partial x_\mu'} = 0 \quad \dots(18)$$

Equations (16) and (18) are the same as equation (14). Thus equations (12) and (13) retain the same form when expressed in terms of primed quantities. Consequently *Maxwell's field equations are invariant under Lorentz transformations.*

16.41. Lorentz Transformations of Electric and magnetic Fields

Maxwell's field equations are

$$\left. \begin{aligned} \text{div } \mathbf{E} &= \frac{\rho}{\epsilon_0} & \dots(a) \\ \text{div } \mathbf{B} &= 0 & \dots(b) \\ \text{curl } \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} & \dots(c) \\ \text{curl } \mathbf{B} &= \mu_0 \mathbf{j} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} & \dots(d) \end{aligned} \right\} \dots(1)$$

The equations when solved for $\mathbf{E}$ and $\mathbf{B}$ in terms of electromagnetic potential $\mathbf{A}$ ϕ yield

$$\mathbf{B} = \text{curl } \mathbf{A} \quad \dots(2)$$

$$\text{and} \quad \mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \text{grad } \phi \quad \dots(3)$$

As Maxwell's field equations are invariant under Lorentz transformations, then in system S' , we must have

$$\mathbf{B}' = \text{curl } \mathbf{A}' \quad \dots(4)$$

$$\text{and} \quad \mathbf{E}' = -\frac{\partial \mathbf{A}'}{\partial t} - \text{grad } \phi' \quad \dots(5)$$

where the primes indicate that the corresponding quantities are functions of the primed coordinates.

The transformation of differential operators, in terms of coordinates (x_1, x_2, x_3, ict) are written as

$$\text{and} \quad \left. \begin{aligned} \frac{\partial}{\partial x_1'} &= \frac{\partial}{\partial x_1} + \frac{v}{c^2} \frac{\partial}{\partial t} \\ \frac{\partial}{\partial x_2'} &= \frac{\partial}{\partial x_2}, \quad \frac{\partial}{\partial x_3'} = \frac{\partial}{\partial x_3} \\ \frac{\partial}{\partial t'} &= \frac{\partial}{\partial t} + v \frac{\partial}{\partial x_1} \end{aligned} \right\} \dots(6)$$

Also the transformations of current four vector $(\mathbf{A}, i\phi/c)$ are given by

$$\left. \begin{aligned} A'_1 &= \frac{A_1 - \frac{v}{c^2} \phi}{\sqrt{1 - \beta^2}}, \quad A'_2 = A_2, \quad A'_3 = A_3 \\ \phi' &= \frac{\phi - v A_1}{\sqrt{1 - \beta^2}} \end{aligned} \right\} \dots(7)$$

To find the transformations of electric and magnetic field vectors $\mathbf{E}$ and $\mathbf{B}$ given by eqn. (4) and (5) we write them in terms of components and then express their R.H.S. in terms of unprimed quantities as :

Equations (4) and (5) in terms of components are written as :

$$\left. \begin{aligned} B_x' &= \frac{\partial A_3'}{\partial x_2'} - \frac{\partial A_2'}{\partial x_3'} \\ B_y' &= \frac{\partial A_1'}{\partial x_3'} - \frac{\partial A_3'}{\partial x_1'} \\ B_z' &= \frac{\partial A_2'}{\partial x_1'} - \frac{\partial A_1'}{\partial x_2'} \end{aligned} \right\} \quad \dots(8)$$

$$\text{and} \quad \left. \begin{aligned} E_x' &= -\frac{\partial A_1'}{\partial t'} - \frac{\partial \phi'}{\partial x_1'} \\ E_y' &= -\frac{\partial A_2'}{\partial t'} - \frac{\partial \phi'}{\partial x_2'} \\ E_z' &= -\frac{\partial A_3'}{\partial t'} - \frac{\partial \phi'}{\partial x_3'} \end{aligned} \right\} \quad \dots(9)$$

Transformations of magnetic field induction components

(i) We have

$$B_x' = \frac{\partial A_3'}{\partial x_2'} - \frac{\partial A_2'}{\partial x_3'} = \frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3} \quad \text{using (6) and (7)}$$

$$\text{i.e.} \quad B_x' = B_x \quad \dots(10a)$$

$$(ii) \quad B_y' = \frac{\partial A_1'}{\partial x_3'} - \frac{\partial A_3'}{\partial x_1'}$$

$$= \frac{\partial}{\partial x_3} \left[\frac{A_1 - \frac{v}{c^2} \phi}{\sqrt{(1 - \beta^2)}} \right] - \frac{\left(\frac{\partial}{\partial x_1} + \frac{v}{c^2} \frac{\partial}{\partial t} \right)}{\sqrt{(1 - \beta^2)}} A_3 \quad (\text{using 6 and 7})$$

$$= \frac{1}{\sqrt{(1 - \beta^2)}} \left[\left(\frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1} \right) - \frac{v}{c^2} \left(\frac{\partial \phi}{\partial x_3} + \frac{\partial A_3}{\partial t} \right) \right]$$

$$= \frac{1}{\sqrt{(1 - \beta^2)}} \left[B_y + \frac{v}{c^2} E_z \right]$$

$$\text{i.e.,} \quad B_y' = \frac{B_y + \frac{v}{c^2} E_z}{\sqrt{(1 - \beta^2)}} \quad \dots(10b)$$

$$(iii) \quad B_z' = \frac{\partial A_2'}{\partial x_1'} - \frac{\partial A_1'}{\partial x_2'}$$

$$\begin{aligned}
 &= \frac{\left(\frac{\partial}{\partial x_1} + \frac{v}{c^2} \frac{\partial}{\partial t}\right) A_2}{\sqrt{(1-\beta^2)}} - \frac{\partial}{\partial x_2} \left[\frac{A_1 - \frac{v}{c^2} \phi}{\sqrt{(1-\beta^2)}} \right] \\
 &= \frac{1}{\sqrt{(1-\beta^2)}} \left[\left(\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2} \right) + \frac{v}{c^2} \left(\frac{\partial A_2}{\partial t} + \frac{\partial \phi}{\partial x_2} \right) \right] \\
 &= \frac{1}{\sqrt{(1-\beta^2)}} \left[B_z - \frac{v}{c^2} E_y \right] \quad \dots(10c)
 \end{aligned}$$

$$\text{i.e.} \quad B_z' = \frac{B_z - \frac{v}{c^2} E_y}{\sqrt{(1-\beta^2)}} \quad \dots(10d)$$

Transformation of electric field strength components

$$\begin{aligned}
 (i) \quad E_x' &= -\frac{\partial A_1'}{\partial t'} - \frac{\partial \phi}{\partial x_1} \\
 &= -\left[\frac{\frac{\partial}{\partial t} + \frac{v}{c^2} \frac{\partial}{\partial x_1}}{\sqrt{(1-\beta^2)}} \right] \left[\frac{A_1 - \frac{v}{c^2} \phi}{\sqrt{(1-\beta^2)}} \right] - \left[\frac{\frac{\partial}{\partial x_1} + \frac{v}{c^2} \frac{\partial}{\partial t}}{\sqrt{(1-\beta^2)}} \right] \times \left[\frac{\phi - v A_1}{\sqrt{(1-\beta^2)}} \right]
 \end{aligned}$$

which on simplification yields

$$\begin{aligned}
 E_x' &= -\frac{\partial A_1}{\partial t} - \frac{\partial \phi}{\partial x_1} \\
 \text{i.e.} \quad E_x' &= E_x \\
 (ii) \quad E_y' &= -\frac{\partial A_2'}{\partial t'} - \frac{\partial \phi'}{\partial x_2'} \quad \dots(11)
 \end{aligned}$$

$$\begin{aligned}
 &= -\left[\frac{\frac{\partial}{\partial t} + \frac{v}{c^2} \frac{\partial}{\partial x_1}}{\sqrt{(1-\beta^2)}} \right] A_2 - \frac{\partial}{\partial x_2} \left[\frac{\phi - v A_1}{\sqrt{(1-\beta^2)}} \right] \\
 &= \frac{1}{\sqrt{(1-\beta^2)}} \left[\left(-\frac{\partial A_2}{\partial t} - \frac{\partial \phi}{\partial x_2} \right) - v \left(\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2} \right) \right] \\
 &= \frac{1}{\sqrt{(1-\beta^2)}} [E_y - v B_z]
 \end{aligned}$$

$$\text{i.e.} \quad E_y' = \frac{E_y - v B_z}{\sqrt{(1-\beta^2)}} \quad \dots(11a)$$

$$(iii) \quad E_z' = -\frac{\partial A_3'}{\partial t'} - \frac{\partial \phi'}{\partial x_3} \quad \dots(11b)$$

$$= -\left\{ \frac{\frac{\partial}{\partial t} + \frac{v}{c^2} \frac{\partial}{\partial x_1}}{\sqrt{(1-\beta^2)}} \right\} A_3 - \left\{ \frac{\partial}{\partial x_3} \frac{\phi - v A_1}{\sqrt{(1-\beta^2)}} \right\}$$

$$\begin{aligned}
 &= \frac{1}{\sqrt{(1 - \beta^2)}} \left[\left(-\frac{\partial A_3}{\partial t} - \frac{\partial \phi}{\partial x_3} \right) + v \left(\frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1} \right) \right] \\
 &= \frac{1}{\sqrt{(1 - \beta^2)}} [E_z + vB_y] \\
 \text{i.e., } E_z' &= \frac{E_z + vB_y}{\sqrt{(1 - \beta^2)}} \quad \dots(11c)
 \end{aligned}$$

The sets of equations (10) and (11) represent the required transformations of magnetic and electric field components.

16.42. Lorentz Force on a Charged Particle

Let S and S' be two systems, the latter moving with velocity v relative to former along positive direction of X -axis. Let a charged particle of charge q be placed in system S' . Thus the charged particle is moving with velocity $v = v\hat{i}$ relative to system S .

As the particle is at rest in system S' , therefore the field in S' is pure electrostatic, i.e., $B_x' = B_y' = B_z' = 0$.

But the particle is moving relative to S , therefore the same field appears to be electromagnetic in system S .

According to Lorentz transformations, the force components transform as

$$F_x' = F_x, F_y' = \frac{F_y}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)}}, F_z' = \frac{F_z}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)}} \quad \dots(1)$$

If $\mathbf{E}'$ is the electric field strength in system S' , then force in system S' is $\mathbf{F}' = q\mathbf{E}'$ (since charge is invariant under Lorentz transformations) or in terms of components

$$F_x' = qE_x', F_y' = qE_y', F_z' = qE_z' \quad \dots(2)$$

Using (1), we get

$$F_x' = F_x = qE_x'$$

$$\left. \begin{aligned} F_y &= F_y' \sqrt{\left(1 - \frac{v^2}{c^2}\right)} = qE_y' \sqrt{\left(1 - \frac{v^2}{c^2}\right)} \\ F_z &= F_z' \sqrt{\left(1 - \frac{v^2}{c^2}\right)} = qE_z' \sqrt{\left(1 - \frac{v^2}{c^2}\right)} \end{aligned} \right\} \quad \dots(3)$$

But according to Lorentz transformations of electric field components, we have

$$E_z' = E_z, E_y' = \frac{E_y - vB_z}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)}} \quad E_z' = \frac{E_z - vB_y}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)}} \quad \dots(4)$$

Using (4), components of force from (3) are written as

$$\left. \begin{aligned} F_x &= qE_x, & \dots (i) \\ F_y &= q(E_y - vB_z) & \dots (ii) \\ F_z &= q(E_z + vB_y) & \dots (iii) \end{aligned} \right\} \quad \dots(5)$$

Multiplying 5 (i), 5 (ii) and 5 (iii) by $\hat{i}$, $\hat{j}$ and $\hat{k}$ respectively and adding, we get

$$\begin{aligned}(\hat{i}F_x + \hat{j}F_y + \hat{k}F_z) &= qE_x \hat{i} + q(E_y - vB_z) \hat{j} + q(E_z + vB_y) \hat{k} \\ &= (\hat{i}E_x + \hat{j}E_y + \hat{k}E_z) + q(vB_y \hat{k} - vB_z \hat{j})\end{aligned}$$

or $\mathbf{E} = q\mathbf{E} + q[v\hat{i} \times (\hat{i}B_x + \hat{j}B_y + \hat{k}B_z)] = q\mathbf{E} + q(\mathbf{v} \times \mathbf{B})$

i.e. $\mathbf{E} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$.

This is required Lorentz force on a particle of charge q .

16.43. Invariants of the Electromagnetic Field

There are two invariants of the electromagnetic field which are

(i) $c^2 B^2 - E^2$

(ii) $\mathbf{E} \cdot \mathbf{B}$

(i) Invariance of $c^2 B^2 - E^2$

According to transformation of electric and magnetic field components

$$\begin{aligned}B'_x &= B_x & E'_x &= E_x \\ B'_y &= \frac{B_y + \frac{v}{c^2} E_z}{\sqrt{1 - \beta^2}} & E'_y &= \frac{E_y - vB_z}{\sqrt{1 - \beta^2}} \\ B'_z &= \frac{B_z - \frac{v}{c^2} E_y}{\sqrt{1 - \beta^2}} & E'_z &= \frac{E_z + vB_y}{\sqrt{1 - \beta^2}}\end{aligned}$$

Therefore,

$$\begin{aligned}c^2 B'^2 - E'^2 &= c^2 B_x'^2 + c^2 B_y'^2 + c^2 B_z'^2 - (E_x'^2 + E_y'^2 + E_z'^2) \\ &= c^2 B_x^2 + c^2 \left[\frac{B_y + \frac{v}{c^2} E_z}{\sqrt{1 - \beta^2}} \right]^2 + c^2 \left[\frac{B_z - \frac{v}{c^2} E_y}{\sqrt{1 - \beta^2}} \right]^2 \\ &\quad - \left[E_x^2 + \left\{ \frac{E_y - vB_z}{\sqrt{1 - \beta^2}} \right\}^2 + \left\{ \frac{E_z + vB_y}{\sqrt{1 - \beta^2}} \right\}^2 \right] \\ &= c^2 B_x^2 + c^2 \left(\frac{B_y^2 + \frac{v^2}{c^4} E_z^2 + 2 \frac{v}{c^2} B_y E_z}{1 - \beta^2} \right) + c^2 \left(\frac{B_z^2 + \frac{v^2}{c^4} E_y^2 - 2 \frac{v}{c^2} B_z E_y}{1 - \beta^2} \right) \\ &\quad - \left\{ E_x^2 + \left(\frac{E_y^2 + v^2 B_z^2 - 2v E_y B_z}{1 - \beta^2} \right) + \left(\frac{E_z^2 + v^2 B_y^2 + 2v E_z B_y}{1 - \beta^2} \right) \right\} \\ &= c^2 B_x^2 + \frac{c^2 B_y^2 + c^2 B_z^2 - E_y^2 - E_z^2}{1 - \beta^2} - \frac{v^2}{c^2} \left(\frac{E_z^2 - E_y^2 + c^2 B_z^2 + c^2 B_y^2}{1 - \beta^2} \right) - E_x^2 \\ &= c^2 B_x^2 + \frac{c^2 B_y^2 + c^2 B_z^2 - E_y^2 - E_z^2}{1 - \beta^2} \left(1 - \frac{v^2}{c^2} \right) - E_x^2\end{aligned}$$

$$= c^2 B_x^2 + c^2 B_y^2 + c^2 B_z^2 - E_x^2 - E_y^2 - E_z^2$$

$$= c^2 \mathbf{B}^2 - \mathbf{E}^2.$$

$$(ii) \mathbf{E}' \cdot \mathbf{B}' = (\hat{i} E'_x + \hat{j} E'_y + \hat{k} E'_z) \cdot (\hat{i} B'_x + \hat{j} B'_y + \hat{k} B'_z)$$

$$= E'_x B'_x + E'_y B'_y + E'_z B'_z$$

$$= E_x B_x + \left\{ \frac{E_y - v B_z}{\sqrt{(1 - \beta^2)}} \right\} \cdot \left(\frac{B_y + \frac{v}{c^2} E_z}{\sqrt{(1 - \beta^2)}} \right) + \left\{ \frac{E_z + v B_y}{\sqrt{(1 - \beta^2)}} \right\} \cdot \left(\frac{B_z - \frac{v}{c^2} E_y}{\sqrt{(1 - \beta^2)}} \right)$$

$$= E_x B_x + \frac{E_y B_y + \frac{v}{c^2} E_y E_z - v B_z B_y - \frac{v^2}{c^2} B_z E_z}{1 - \beta^2}$$

$$+ \frac{E_z B_z - \frac{v}{c^2} E_y E_z + v B_y B_z - \frac{v^2}{c^2} B_y E_y}{1 - \beta^2}$$

$$= E_x B_x + \frac{E_y B_y + E_z B_z}{1 - \beta^2} \left(1 - \frac{v^2}{c^2} \right)$$

$$= E_x B_x + E_y B_y + E_z B_z = \mathbf{E} \cdot \mathbf{B}. \quad \dots (2)$$

From equation (1) and (2) it is obvious that $c^2 \mathbf{B}^2 - \mathbf{E}^2$ and $\mathbf{E} \cdot \mathbf{B}$ are invariant under Lorentz transformations. The invariance of these quantities indicates that the electromagnetic field of a plane wave appears similar in all inertial frames.

16.44. General Theory of Relativity

The two fundamental theories in physics which have been of great importance in studying the behaviour of matter are :

(i) Newtonian theory of gravitation—which describes the behaviour of one mass point on the other and

(ii) The electrodynamics—which describes the behaviour of charged matter in the presence of electromagnetic fields.

The special theory of relativity had its origin in the development of electrodynamics, while the general theory of relativity is the relativistic theory of gravitation.

The special theory of relativity only accounts for inertial systems, in the region of free space, where gravitational effects can be neglected. In those systems the law of inertia holds good and the physical laws retain the same form. The special theory of relativity does not account for non-inertial (i.e., accelerated) systems. For example the 'clock paradox' and 'universal phenomenon of gravitation' could not be accounted by special theory of relativity. Thus naturally we wish to extend the principle of relativity in such a way that it may hold even for non-inertial systems and consequently the extended theory may explain the non-inertial phenomena like clock paradox and particularly the phenomenon of gravitation. The extended theory is known as the *general theory of relativity*. In developing the general theory of relativity it is helpful to analyse the predictions of special theory of relativity with respect to the phenomenon of gravitation. On the generalisation of special theory of relativity with respect to gravitational phenomenon, the theoretical predictions led to small deviations from the observed phenomenon in the following cases :

(1) **Advance of Perihelion of Planets** : The special theory of relativity of gravitation leads to the precession of the perihelion of planets; but the precession of a planet observed and that accounted by special theory of relativity are not in the same amounts. For example in the

case of planet mercury the special theory of relativity accounts a retardation of perihelion at the rate of 7.2 seconds of arc per century; while observations show that the perihelion of mercury advances at the rate of 43 seconds of arc per century; thus the effect observed is six times greater in magnitude as accounted by special theory of relativity.

(2) **Shift in spectral lines** : The special theory of relativity of gravitation predicts no shift of spectral lines emitted by atoms even in powerful gravitational fields; while observations indicate a shift of spectral lines towards red.

(3) **Deflection of light rays due to gravitational field** : The special theory of relativity predicts that when the light rays pass close to the sun, they are deflected by an amount 0.88 second of an arc; but observations show that the deflection is 1.75 seconds of arc; which is twice the result of special theory of relativity of gravitation.

The special theory of relativity, due to these deviations from observations, may not be considered wrong since for low velocities it reduces to Newtonian theory which is valid to a high degree of accuracy. However the special theory of relativity of gravitation seems to be slightly wrong for the phenomenon in which the velocity approaches the velocity of light, because it could not account the correct amount of bending of light rays towards the sun. The special theory of relativity of gravitation seems to fail for fixed particles in the gravitational fields; because it could not predict the red shift of spectral lines emitted by atoms even in powerful gravitational fields. The special theory of relativity also seems to be wrong for the phenomenon when velocity and gravitational field both are present since it could not predict correctly the process of perihelion of planets which is considered to be due to their velocity and the strength of gravitational field.

Thus it is obvious that Minkowskian space-time continuum does not account the natural phenomenon of gravitation in non-inertial frames. This is also due to the fact that in general theory of relativity we consider that the motion in gravitational field is some type of inertial motion as accelerating the passengers relative to a suddenly stopped carriage. The acceleration is due to Newton's law of inertia. "A body at rest remains at rest and a body in motion moves with the same velocity till no external force is applied." This also explains why the acceleration due to action of gravity is independent of the mass of the body. As the force of gravitation is zero at infinite distance from the mass or attracting bodies; therefore it is assumed the space close to the attracting bodies or the mass does not follow the Minkowskian space-time continuum. Hence for accounting the observed facts of gravitation we have to modify the Minkowski space-time continuum. The observations of gravitational red shift on earth support the modification in Minkowskian space-time structure.

In Minkowski space-time continuum the line element is given by

$$ds^2 = -dx^2 - dy^2 - dz^2 + c^2 dt^2 \quad \dots(1)$$

$$(\text{or } = dx^2 + dy^2 + dz^2 - c^2 dt^2)$$

The modified form of above equation in tensor form is

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \quad \dots(2)$$

Obviously eqn. (2) may reduce to (1) in special case. Thus in regions far from gravitational field the general theory of relativity reduces to special one and the Minkowski space-time continuum holds in regions far from gravitational field.

The element in (2) represents the curved geometry. Thus according to general theory of relativity the space is curved in a gravitational field. Since the space is curved in gravitation field, therefore the geometry of space in a gravitational field is Riemannian, for the idea was originally developed by Riemann. Thus free motion in a gravitational field or in a non-Euclidian space is not straight; but curvilinear.

Thus if we consider a very small element of this curve, it will be nearly a straight line, therefore it is very difficult to distinguish the gravitational and inertial forces in small regions

of space. In such regions a non-inertial co-ordinate system is equivalent to inertial system in which an additional gravitational field operates with the same acceleration of falling bodies which is due to inertial forces in the non-inertial systems; since classically the difference between the inertial and non-inertial systems is merely that in the latter the inertial forces come into play.

Hence the theory of gravitation also deals with non-inertial systems unlike special theory which deals only inertial systems; due to this reason *the theory of gravitation is called "general theory of relativity."*

16.45(a) Principle of General Covariance

We know that the special theory of relativity is limited only for inertial systems. According to it the principle of relativity is "The laws of nature retain the same form in all inertial systems." For generalising the above principle for all coordinates systems, inertial as well also non-inertial, we have simply to remove the word inertial. Thus the generalised principle of relativity states,

"The laws of nature retain their same form in all coordinate systems." In other words *the laws of nature remain covariant independent of the frame of reference.*

This statement is called the **principle of general covariance**. According to this principle we must express all the physical laws of nature by means of equations in the covariant form, which are independent of the coordinate systems. This can be done by expressing the laws of nature in the form of tensor equations; because a tensor equation has exactly the same form in all co-ordinate systems.

The tensor form of the line element is

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \quad (\mu, \nu = 1, 2, 3, 4) \quad \dots(1)$$

where $g_{\mu\nu}$ is a fundamental covariant tensor of rank 2 obeying the transformation law

$$\bar{g}_{ij} = \frac{\partial x^\mu}{\partial \bar{x}^i} \frac{\partial x^\nu}{\partial \bar{x}^j} g_{\mu\nu}$$

where the quantities carrying bar correspond to the new coordinate system.

As an example that the tensor form of a law follows that general covariance principle, consider that a certain law of nature in system of variable x is represented by the tensor equation

$$A_\nu^\mu = B_\nu^\mu \quad \dots(2)$$

Then this law when transformed to new system of variables $\bar{x}$ may be written as

$$\begin{aligned} \bar{A}_\nu^\mu - \bar{B}_\nu^\mu &= \frac{\partial \bar{x}^\mu}{\partial x^{ji}} \frac{\partial x^j}{\partial \bar{x}^\nu} A_j^i - \frac{\partial \bar{x}^\mu}{\partial x^i} \frac{\partial x^j}{\partial \bar{x}^\nu} B_j^i \\ &= \left(A_j^i - B_j^i \right) \frac{\partial \bar{x}^\mu}{\partial x^i} \frac{\partial x^j}{\partial \bar{x}^\nu} \\ &= 0 \quad [\text{since } A_j^i = B_j^i \text{ from (1)}] \end{aligned}$$

$$\text{i.e.,} \quad \bar{A}_\nu^\mu = \bar{B}_\nu^\mu \quad \dots(3)$$

Equation (3) has exactly the same form as eqn. (2). Thus the laws of nature, when expressed in the form of tensor-equations, follow the general covariance principle. Hence according to general covariance principle the laws of nature must be expressed in the form of tensorial equations.

16.45(b). Principle of Equivalence

According to principle of covariance the physical laws must be expressed in the form of covariant tensor equations; so that they may remain unchanged in all coordinate systems. But in order to introduce the effects of gravitational field in the relativistic theory of gravitation, Einstein gave another principle known as the *principle of equivalence*.

Consider two systems S and S' , the latter moving with uniform acceleration a_0 relative to the former. If we consider the first frame S to be inertial, then second frame S' will be non-inertial and vice-versa. Consider that a body of inertial mass m not acted by any force is at rest in inertial system S . Then there is no force acting on the body relative to system S , but if observed from system S' , the body appears to experience a force $-ma_0$. Obviously the force $-ma_0$ acting on the body is not real and arises as a result of the acceleration of the non-inertial frame. Such a force which appears only due to acceleration of the non-inertial frame is called the *fictitious* or *apparent* or *inertial force*. The accelerations which do not correspond to real forces acting on the body; but give rise to inertial forces are called inertial accelerations. When the non-inertial frame rotates uniformly about an axis fixed with respect to the inertial frame, the inertial acceleration is the centrifugal acceleration. Thus if we refer to motion of the body to a non-inertial system, we come across inertial acceleration which is evidently independent of the mass of the body. Thus in accelerated system we always get inertial force. In order to introduce the effects of gravitational action Einstein pointed out that the inertial acceleration is similar to gravitational acceleration. For example a person, in an elevator feels to become momentarily heavier when the elevator is accelerated upward and lighter when the elevator is accelerated downward. This example of daily experience initiates closely to an actual change in the force of gravity. Thus the classical picture of a gravitational field is artificial, since it is possible for an observer not to detect such a field at all by choosing a suitable frame of reference. Consequently Einstein pointed out that a gravitational field can be produced by accelerating uniformly an inertial frame of reference. By this analogy Einstein gave the *principle of equivalence* which states. "*In the neighbourhood of any given point it is not possible to distinguish between the gravitational field produced by the attraction of masses and the field produced by accelerating an inertial frame of reference.*" Thus we learn from this principle that the two fields, a gravitational field produced by attraction of matter in the universe and the other field produced by accelerating uniformly an inertial frame outside all gravitational field, are identical provided the acceleration of the inertial frame is equal and opposite to the actual gravitational acceleration. Obviously according to the principle of equivalence the gravitational and inertial forces are of the same nature and obey the same laws.

The principle of equivalence may be illustrated by the following simple example :

Consider an enclosure, from which nothing can be known of what goes on outside it, fixed on the earth. Then the observer in the enclosure would observe that :

- (i) The bodies possess weight.
- (ii) All the bodies left to themselves fall in the same way with an acceleration of gravity, g irrespective of their masses.
- (iii) A horizontal projectile traverses a parabola.

Now let us imagine that the same enclosure is falling freely under the action of gravity. Now the observer in the enclosure would observe that :

- (i) The bodies are weightless.
- (ii) All the bodies left to themselves float just at that place and do not fall anywhere.
- (iii) A horizontal projectile traces a straight line,

i.e., everything takes place as in the absence of gravitational field. In other words all traces of gravitational field would have disappeared for the observer in the freely falling enclosure. **Thus**

gravitational field can be abolished or suppressed by a system such as the enclosure having a proper uniform acceleration.

Conversely suppose that the same enclosure be placed far from all massive bodies so that the enclosure is *outside all gravitational field*. Now the observer in the enclosure would observe that :

- (i) The bodies are weightless.
- (ii) All the bodies left to themselves float just at that place and do not fall anywhere.
- (iii) A horizontal projectile traces a straight line,

i.e., everything takes place as in the absence of gravitational field. Now imagine that the enclosure is pulled up by a rope and kept in uniform accelerated motion, the upward acceleration being equal in magnitude to that of gravity, *i.e.*, $|a| = |g|$; then the observer in the enclosure would observe :

- (i) The bodies possess weight equal to that in earth's gravitational field.
- (ii) All the bodies left to themselves falling the same way with an acceleration equal to that of gravity irrespective of their masses.
- (iii) A horizontal projectile traces a parabola.

i.e., everything takes place as if he were in earth's gravitational field. Thus the motion of the bodies cannot be distinguished in the two cases : (a) when the enclosure is in earth's gravitational field and (b) when the enclosure outside all gravitational field, is accelerated upward with an acceleration equal to that of gravity. Thus, **it is possible to produce a gravitational field by accelerating properly an inertial frame uniformly.**

Thus we see that a gravitational field can be produced or abolished according to our wish by choosing a uniformly accelerated proper frame of reference. Summarizing above observations we come to the following general conclusion :

"A system which is stationary in a gravitational field of strength g is physically equivalent to a system which is in gravitational free space but accelerated in the opposite direction with an acceleration of g " which is the principle of equivalence.

According to principle of equivalence a uniform field can be replaced by a single system of reference. Hence all frames of reference become equally suitable for the description of physical laws. The principle of equivalence is true for electrical and optical phenomena as well. For example if a ray of light follows a rectilinear path with respect to an inertial frame of reference S' , it is found that the same ray of light no longer follows the rectilinear path with respect to another system of reference S'' , which is an accelerated motion. From this it follows that **the rays of light, in general, follows the curvilinear paths in gravitational fields.**

The principle of equivalence is a very powerful tool in the general theory of relativity.

EXERCISES

Short Answer Questions

1. What do you mean by a frame of reference ?
2. What do you mean by inertial and non-inertial frames of reference ?
3. Is earth an inertial frame ? If not why ?
4. State Galilean transformations for space and time.
5. What is the path of one projectile as seen from another projectile ? [Ans. *Straight line*]
6. With what aim, the Michelson Morley experiment was performed ? State its conclusions.
7. State observational evidences in support of constancy of speed of light in different inertial frames.

8. State basic postulates of special theory of relativity.
9. State Lorentz transformation equations. Why Lorentz transformations are superior to Galilean transformation ?
10. What is Lorentz-Fitzgerald contraction ?
11. What do you mean by time dilation ?
12. Why does mass of a body vary with velocity ? State formula of mass-variation with velocity.
13. Express Newton's equation of motion in relativistic form.
14. What do you mean by mass-energy equivalence relation ?
15. What is the maximum possible velocity of a material particle ?
16. What is the rest mass of a photon ?
17. Prove the relation $E^2 - p^2c^2 = m_0^2c^4$ where symbols have their usual meanings.
18. Write relativistic Lagrangian. Is it equal to $T - V$ as in classical Lagrangian ?
19. Write relativistic Hamiltonian. Does it represent $H = T + V$ as in Classical Hamiltonian ?
20. State velocity addition theorem in relativity.
21. What is Minkowski's space ?
22. What do you mean by space like and time like intervals ?
23. What is Lorentz condition ? Express it in covariant form.

Long Answer Questions

1. (a) What are the inertial frames ? How has it been established that earth is not an inertial frame? (Agra 1986)
(b) Explain non-inertial from of reference with suitable examples (Rohilkhand 2000)
2. Derive Galilean transformation equations and discuss the theorem of conservation of linear momentum.
(a) What are the Galilean transformations for the velocity of a particle ?
(b) Show that a frame of reference having a uniform translatory motion relative to an inertial frame is also inertial. (Rohilkhand 1986)
4. Show that the motion of one projectile as seen from another projectile will always be a straight line motion. (Kanput 2005, 1999, Agra 1995; Rohilkhand 2004, 2001, 1991, 88)
5. What are Galilean transformations ? Obtain the transformatory equation for velocity and acceleration in frame S' moving with velocity v relative to inertial frame S . (Agra 1999)
6. What are Galilean transformations ? Use the transformations to show that the distance between two points is invariant in two inertial frames. (Rohilkhand 2005)
7. Show that the law of conservation of linear momentum can be obtained from the law of conservation of energy and the principle of Galilean invariance. (Rohilkhand 2002, 90)
8. Show that the basic laws of physics are invariant under Galilean transformations.
9. Discuss the Michelson-Morley experiment. Explain how the result obtained is interpreted by the principle of relativity. (Banaras 1988)
10. (a) What is an inertial frame ? Prove that in all material frames connected by Lorentz transformations the fundamental velocity is the same.
(b) Explain what is meant by the null results of the Michelson-Morely experiment, How does it lead to the conclusion that the velocity of light in free space is the fundamental velocity ? (Banaras 1990)
11. Explain Michelson-Morely experiment. Show how the null result of the experiment was explained. (Ranchi 2004, Rohilkhand 2005, 2004, 2001, 1994, 92)
12. Describe the Michelson and Morley experiment and discuss its importance in the evolution of the theory of relativity. (Garhwal 2003, 1999, Agra 2003, 1998)

13. (a) State the fundamental postulates of special theory of relativity and deduce from them the Lorentz transformations and show how these are superior to Galilean transformations.
(Ranchi 2004, Garhwal 1998, Rohilkhand 2000, Meerut 2000, Nagpur 1998, Agra 97)
- (b) Discuss simultaneity of events in two inertial frames of reference in the light of above transformations.
(Nagpur 1987)
14. Write down Lorentz transformation equations and hence explain Lorentz Fitzgerald contraction and time dilation.
(Agra 1997)
15. (a) Explain (i) length contraction and (ii) time dilation in special relativity, what are proper length and proper time interval.
(Garhwal 1999)
- (b) Explain why we can not accelerate an electron to a velocity greater than the velocity of light in free space, Can it be done in material medium?
(Garhwal 1991)
16. Define proper time interval Explain why a moving clock appears to go slow to a stationary observer.
(Rohilkhand 1993)
17. What do you understand by time dilation? What is the proper interval of time? Establish a relation between proper and non-proper time intervals.
18. Discuss the concept of space and time in special relativity theory. Prove that the physical laws retain their validity under Lorentz transformations.
19. Using Lorentz transformation equation, discuss the concept of time dilation and length contraction.
(Patna 2004)
20. Using Lorentz transformations, obtain the law of addition of velocities. From this, show that a photon moving within velocity 'c' in one system S will appear to move with the same velocity 'c' in another system S' which is moving with a velocity relative to S.
(Rohilkhand 1989)
21. Show on the basis of relativistic addition theorem of velocities that the velocity of light is an absolute constant independent of the frame of reference and that it is the maximum velocity attainable in nature.
(Kanpur 2001, Agra 1999)
22. Mention a few consequences of Lorentz transformations. Derive the law of velocity transformation and show that c subtracted from c leads to c itself.
(Garhwal 1990)
23. Derive an expression for the variation of mass with velocity. Find the velocity of a particle at which mass of the particle be double its rest mass.
(Patna 2004)
24. A body moving with velocity v has mass m. Show that $m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$ where m_0 is rest mass of the body and c, the speed of light.
(Rohilkhand 2000)
25. (a) What do you understand by relativistic length contraction.
- (b) Explain and establish mass energy equivalence $E = mc^2$ Give its two physical examples
(Garhwal 1998, Rohilkhand 1994, 88, 82, Agra 1996)
26. State the postulates of the special theory of relativity and based on these, obtain Lorentz as well as inverse Lorentz transformations. Hence obtain an expression to conclude that a moving clock runs more slowly than a stationary clock.
27. What are consequences if momentum and energy are invariant under Lorentz transformations? Prove the relation $E^2 - p^2c^2 = m_0^2 c^4$, where symbols have their usual meaning.
28. What is relativistic energy? Prove the relation $E^2 - p^2c^2 = m_0^2 c^4$ where p is the relativistic momentum.
Derive an expression for the velocity of a particle in terms of its relativistic momentum and energy.
29. Discuss the equivalence of mass and energy, deriving the expression. Explain it by giving some examples.
(Patna 2004, Rohilkhand 1989, 84)
30. Derive the relativistic relation for variation of mass with velocity and obtain from it the mass energy relation $E = mc^2$ (Rohilkhand 2001, Garhwal 1999, Bombay 1976, Agra 1976)

31. (a) Write down the momentum energy transformations.
 (b) Show that the rest mass of a particle of momentum p and kinetic energy T is
- $$m = \frac{p^2 c^2 - E^2}{2T c^2}$$
32. (a) Assuming the invariance of equations of light wave front in two inertial frames moving with uniform relative velocity, derive the Lorentz transformations.
 (b) Show that the length of a moving rod appears to be contracted.
33. What is Doppler's effect in light ? Obtain a relativistic formulation of Doppler's effect. What do you mean by Red shift ? (Patna 2004)
34. Obtain relativistic expression for (a) Doppler's effect, (b) Aberration of star light.
35. What is MINKOWSKI 4 - D Space ? What is the necessity of introducing time dimension. (Garhwal 1996)
36. Prove that Lorentz transformation in Minkowski space is an orthogonal transformation. (Kerala 2009, 2002)
37. Distinguish between space like and time like world points in terms of their connectivity by light signals. (Kerala 2002)
38. Explain a four vector. Discuss Minkowski four-dimensional formulation bringing out the significance of the fourth component of momentum and equation of motion. (Meerut 2004, 2001)
39. Show that the Lorentz transformations can be regarded as transformations due to rotation of axes in the four-dimensional Minkowski space. Discuss the relativity of simultaneity. (Meerut 2003)
40. Discuss Minkowski four-dimensional space. Write velocity and momentum as four vectors and hence write Lorentz transformations in four-dimensional space. Discuss the conservation of four-momentum. (Meerut 2002)
41. Define a four-vector. How are the components of the four momentum vector related to the three-momentum of a particle ? (Meerut 2010, Purvanchal 2004, Meerut 1998, Kanpur 2005, Rohilkhand 2003)
42. Discuss Lagrangian formulation of relativistic mechanics. (Meerut 1999)
43. Derive relativistic Lagrangian and Hamiltonian for a free particle.
44. Show that the Maxwell's electromagnetic equations are invariant under the Lorentz transformation. (Meerut 1998; Vikram 1989; Agra 2000)
45. If two reference frames are related to each other by the Lorentz transformations

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}, y' = y, z' = z = \frac{t - \frac{vx}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

how are electromagnetic potential (ϕ, A) and electromagnetic field strength (E, H) in two systems related ? (Agra 1985)

46. State Maxwell's equations and show that they are invariant under Lorentz transformations. (Kanpur 2004, 1988)
47. Express Maxwell's equations in covariant form and derive the transformation law for the electric and magnetic fields. (Kanpur 2005, 1992; Agra 2002, 1999)
48. The relativistic Lagrangian for a charged particle in a constant magnetic field is given by

$$L = -mc^2 \sqrt{1 - \beta^2} + \frac{v}{c} A \cdot v$$

where $A = \frac{1}{2} (A \times r)$

- Show that the Lagrangian leads to the Lorentz force on the particle. (Kerala 2002)
49. (a) Using the Lorentz transformation equations for the electric and magnetic fields, obtain the fields of a point charge q moving in a straight line path with a velocity v .
 (b) Show that $c^2 B^2 - E^2$ is an invariant under Lorentz transformations. What is its form in four dimensional notation? (Meerut 2002, 1998)
50. (a) Define the field strength tensor and write its components explicitly. Show how Maxwell's equations can be written in covariant form.
 (b) Determine the Lorentz transformation equations for the field vectors $\mathbf{E}$ and $\mathbf{B}$ and show that the scalar product $\mathbf{E} \cdot \mathbf{B}$ is unchanged under the Lorentz transformations. (Meerut 2005, 2001, 1994)
51. (a) Show that the Lorentz force on a single charged particle moving in an electromagnetic field with velocity $\mathbf{v}$ is given by

$$\mathbf{f} = q [\mathbf{E} + \mathbf{v} \times \mathbf{B}]$$

where the symbols have their usual meaning.

- (b) Derive the transformation equations for $\mathbf{E}$ and $\mathbf{B}$. (Agra 2005, 1997)

$$\tan \theta = \frac{\sin \theta' \sqrt{1 - \frac{v^2}{c^2}}}{\cos \theta' + \frac{v}{c}}$$

Can be derived from the velocity transformation relations. (CSE 2006)

Problems

- A rod has a length of one metre. It is placed in a spaceship moving with a velocity $0.4c$ relative to the earth. Calculate its length as measured by an observer (i) in spaceship (ii) on earth. (Rohilkhand 1996)
 [Ans. (i) 1 m (ii) 0.92 m]
- A rod of 10 metre length is moving to earth at a speed of $0.92c$ (c = speed of light in free space). What would be the length of the rod as seen by an observer moving with the rod? (Rohilkhand 1984)
 [Ans. 10 m]
 [Hint. Rod is at rest relative to observer].
- In the laboratory the life time of a particle moving with speed 2.8×10^8 m/s, is found to be 2.5×10^{-7} s. Calculate the proper life time of the particle. (Agra 1969)
- What is the velocity of nuclear particles whose mean life time is 2.5×10^{-7} seconds? The proper life time is 2.5×10^{-8} sec. (Rohilkhand 1999, Agra 1997)
- What is velocity of π mesons whose proper mean life is 2.5×10^{-8} sec and observed mean life is 2.5×10^{-7} sec.? $\left[\text{Ans. } \frac{3c}{\sqrt{10}} = 2.84 \times 10^8 \text{ m/s} \right]$
- In the laboratory two particles are observed to travel in opposite directions with speed 2.8×10^8 m/s. Deduce the relative speed of the particles. (Agra 1996)
 [Ans. 2.994×10^8 m/s]
- What will be the apparent length of a metre stick measured by an observer at rest, when the stick is moving along its length with a velocity equal to
 (i) c (ii) $\frac{c}{\sqrt{2}}$ (iii) $\frac{\sqrt{3} c}{2}$ (iv) $\frac{c}{2}$

- [Ans. (i) 0, (ii) 0.70m, (iii) 0.5 m, (iv) 0.869 m]
8. The length of space-ship is measured to be exactly half its proper length.
 (a) What is the speed of the space-ship relative to observer ?
 (b) What is the dilation of spaceship's unit time ?
 [Ans. (a) 0.866c, (b) rate of clock will slow to half]
9. A scientist observes that a certain atom A moving relative to him with velocity 2×10^8 m/s, emits a particle B which moves with velocity 2.8×10^8 m/s, with respect to atom, calculate the velocity of the emitted particle relative to scientist. [Ans. 2.95×10^8 m/s.]
10. A certain particle has a life time of 10^{-7} second, when measured at rest. How far does it go before decaying if its speed is $0.99c$ when it is created ? [Ans. 208.97 m]
11. An unstable particle has a life-time of $5 \mu\text{s}$ in its own frame of reference and is moving towards the earth with a speed $0.8c$. What will be the life time of the particle on the earth ?
 [Ans. $8.3 \mu\text{s}$] (CSE 2007)
12. (a) Write down the relativistic expression for kinetic energy of a body and show that for small speeds it reduces to the classical expression.
 (b) Deduce the minimum energy of gamma-ray photons in MeV which can cause electron position pair-production ($m_0 = 9.1 \times 10^{-31}$ kg) (Rohilkhand 1998, Agra 1992)
 [Ans. 1.02 MeV]
13. State the formula for variation of mass with velocity. Compute to three significant places the speed at which the mass of a body becomes 8 times of that at rest ($c = 3.00 \times 10^8$ m/s).
 (Meerut 1992)
 [Ans. 2.98×10^8 m/s]
14. Compute the speed at which mass of an electron becomes 4 times its rest mass. What is the physical significance of this increase of mass ?
 (Meerut 1989)
 [Ans. 2.94×10^8 m/s]
15. It is assumed that the sun gets energy by fusion of 4 hydrogen atoms to a helium atom. Rest mass of hydrogen and helium are 1.0081 and 4.0039 atomic mass units respectively. Calculate the energy released in process [$1 \text{ a.m.u} = 1.66 \times 10^{-27}$ kg.] [Ans. 26.53 MeV]
16. A red hot sphere of iron (specific heat capacity $0.11 \text{ cal./g. } ^\circ\text{C}$ weighs 1 kg. If the sphere cools to 1200°C , find the equivalent loss in mass. [Ans. 5.6×10^{-12} kg.]
17. A hundred μ mesons each of rest mass 206 electron and energy 4.75 BeV are produced at an altitude of 30 km. If the mean life of μ mesons at rest is 2.2×10^{-8} s calculate their number expected to reach the sea level.
 (a) allowing for time dilation.
 (b) Neglecting time dilation.
 Take the electrons to travel vertically downwards without loss of energy. Give rest mass as 0.5 MeV. What conclusion can you draw from your result. ?
 [Ans. (a) = 38, (b) = 1.7×10^6]
18. An inertial frame of reference is moving with respect to other with uniform velocity v along y -direction. Write down Lorentz transformation equations. Hence prove that three dimensional volume element ' $dx dy dz$ ' is not invariant; but four dimensional volume element ' $dx dy dz dt$ ' is invariant under Lorentz transformations. (Garhwal 1985)
19. Calculate the velocity of a particle at which its mass will become five times its rest mass. (Rohilkhand 1985)
20. Show that the circle $x^2 + y^2 = a^2$ in frame S appears to be an ellipse in frame S' which is moving with velocity v relative to S . (Kanpur 1997, Rohilkhand 1985)
 [Hint. Equation of circle in frame S is $x^2 + y^2 = a^2$]

According to length contraction $x = x' \left(1 - \frac{v^2}{c^2} \right)^{-1/2}$; $y = y'$

So
$$x'^2 \left(1 - \frac{v^2}{c^2} \right)^{-1} + y'^2 = a^2$$

i.e.,
$$\frac{x'^2}{a^2 \left(1 - \frac{v^2}{c^2} \right)} + \frac{y'^2}{a^2} = 1$$

which is the equation of ellipse.]

21. With what velocity should a clock be moved relative to an observer so that it may seem to lose 1 minute per day (Garhwal 1997,96)

[Ans. 1.12×10^7 m/s]

22. An electron and a positron at rest come together and annihilate each other. Calculate the energy exactly released. (Rohilkhand 2000)

[Ans. 1.02 MeV]

23. Calculate the lengths and the orientation of a rod of length 5m in a frame of reference moving with velocity $0.6c$ in a direction making angle of 30° with the rod. (Rohilkhand 1999, 95)

[Ans. 4.27 m, $\theta' = 35.8^\circ$]

Multiple Choice Questions : Select the right Choice

1. An inertial frame is

(a) accelerated (b) decelerated
(c) unaccelerated (d) may be accelerated or unaccelerated

(Agra 2008)

2. Earth is

(a) an inertial frame
(b) a non-inertial frame
(c) inertial frame at day and non-inertial at night
(d) inertial frame at night and non-inertial at day.

3. Michelson-Morley experiment proved that

(a) earth is an inertial frame (b) earth is a non-inertial frame
(c) earth is absolute frame (d) speed of light is same in all inertial frames.

4. Consider the following statements :

1. Michelson-Morley experiment indicate that the speed of light is invariant.
2. The stationary ether hypothesis cannot explain the results of Michelson-Morley experiment.
3. Galilean transformations are good for electromagnetism, but not for mechanics. Which of the above statements is/are true ?

(a) 1, 2, and 3 (b) 1 and 2 only (c) 2 and 3 only (d) 1 and 3 only.

5. In Galilean transformations, time interval is

(a) different for different frames (b) a vector
(c) relative (d) same for all frames.

6. A body of charge q starts moving with velocity $0.8c$ (c being speed of light in vacuum). The charge on body in motion is

(a) q (b) $\frac{q}{2}$ (c) $0.6q$ (d) $0.8q$.

7. A body of rest mass m_0 starts moving with velocity $0.8c$ (c being speed of light in vacuum). The mass of the body in motion is

(a) m_0 (b) $\frac{m_0}{2}$ (c) $0.6m_0$ (d) $1.67m_0$.

8. A rod of proper length l_0 starts moving with velocity $0.8c$. The length of rod in motion is
 (a) l_0 (b) $\frac{l_0}{2}$ (c) $0.6l_0$ (d) $1.67l_0$
9. The specific charge of β -rays is found to be less than that of cathode rays. This is because
 (a) charge has decreased by virtue of large velocity
 (b) mass has decreased by virtue of large velocity
 (c) mass has increased by virtue of large velocity
 (d) charge has decreased and mass has increased by virtue of large velocity.
10. A metre stick moves along the length with a relativistic velocity v . Then the stick
 (a) extends along its length (b) contracts along its length
 (c) contracts perpendicular to length (d) contracts in all directions.
11. How will a square object moving with relativistic speed $0.6c$ appear to a stationary observer? (CBSE 2008)
 (a) Square (b) Rectangle (c) Triangle (d) Circular.
12. A cubical block of each side l_0 moves with velocity v directed along one side. The volume of cube as observed from a stationary observer, is
 (a) $\frac{l_0^3}{\sqrt{1 - \frac{v^2}{c^2}}}$ (b) $l_0^3 \sqrt{1 - \frac{v^2}{c^2}}$ (c) $l_0^3 \left(1 - \frac{v^2}{c^2}\right)^{3/2}$ (d) $l_0^3 \left(1 - \frac{v^2}{c^2}\right)^{-3/2}$
13. A cube of edge-length a is kept stationary in a fixed frame. A second frame is moving parallel to one of the edges of the cube. An observer in the moving frame observes that the volume of the cube is one-half of the value in the fixed frame. What is the speed of the moving frame? (CSE 2006)
 (a) c (b) $c/2$ (c) $\sqrt{3}c/2$ (d) $\frac{2}{3}c$
14. A cube of side 3 m is at rest. What is the approximate volume of the cube, when it moves with a velocity $v = 2 \times 10^8\text{ m/s}$, parallel to one of its sides? (CSE 2007)
 (a) 20.2 m^3 (b) 14 m^3 (c) 19 m^3 (d) 27 m^3
15. A metal block of density d moves with speed $0.6c$. Then its density
 (a) increases
 (b) decreases
 (c) remains unchanged
 (d) may decrease or increase depending on direction of velocity.
16. A cube has density ρ_0 , when at rest. What is the density of the cube when it moves with a velocity v (close to c) parallel to one of its sides? (CSE 2005)
 (a) $\rho_0 \left(1 - \frac{v^2}{c^2}\right)^{1/2}$ (b) $\rho_0 \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$ (c) $\rho_0 \left(1 - \frac{v^2}{c^2}\right)^{-1}$ (d) $\rho_0 \left(1 - \frac{v^2}{c^2}\right)$
17. By what factor does the density of an object change when it moves with velocity v .
 (a) $\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$ (b) $\frac{1}{1 - \frac{v^2}{c^2}}$ (c) $\sqrt{1 - \frac{v^2}{c^2}}$ (d) $\frac{v}{c}$
18. A young lady of 25 years starts running with relativistic speed v ; then she would appear to her stationary friends
 (a) thin and younger (b) fat and older
 (c) fat and younger (d) thin and older.
19. According to theory of relativity Newtonian Mechanics is true
 (a) approximately for all velocities (b) approximately for low velocities
 (c) approximately for high velocities (d) exactly for all velocities.

20. Lorentz transformations assume
 (a) Space and time are both relative (b) Space is relative but time is absolute
 (c) Space is absolute but time is relative (d) Space and time both are absolute.
21. According to mass-energy equivalence, the unit of mass may be expressed as
 (a) joule (b) MeV (c) MeV/c (d) MeV/c²
22. In pair production
 (a) mass is converted into energy (b) energy is converted into mass
 (c) two photons are produced (d) two electrons are annihilated.
23. In pair annihilation
 (a) mass is converted into energy (b) energy is converted into mass
 (c) only one photon is produced (d) two electrons are annihilated.
24. Doppler's effect in sound depends on whether the source or observer or both have relative motion, while Doppler's effect in light depends only on relative velocity of source and observer. The reason for this is
 (a) Einstein-mass energy relation (b) photoelectric effect
 (c) invariance of speed of light (d) high speed of light.
25. A space ship is travelling with a velocity $0.4c$, where c is velocity of light. A person performing an experiment in the spaceship observes a particle moving with velocity $0.4c$ in the same direction as that of the motion of the spaceship. A stationary observer on the earth would observe the particle to have the velocity
 (a) $0.69c$ (b) $0.50c$ (c) $0.80c$ (d) $0.73c$.
26. Two particles approach each other with speed $0.8c$ with respect to laboratory. Their relative speed will be
 (a) c (b) $1.8c$ (c) $0.976c$ (d) zero.
27. The percentage contraction of a rod moving with velocity $0.8c$ in the direction inclined at 60° to its own length is
 (a) 6% (b) 9% (c) 40% (d) 60%.
28. The length of a space-ship is measured to be exactly half its proper length, then space-ship's unit time will be
 (a) slow down to half (b) go twice as fast
 (c) unchanged (d) slow down to one-fourth.
29. If the relativistic mass of a particle is twice its rest mass, what is the ratio of its speed to that of light ? (CSE 2008)
 (a) $\frac{\sqrt{3}}{2}$ (b) $\frac{1}{\sqrt{2}}$ (c) $\frac{1}{2}$ (d) $\frac{1}{4}$
30. The kinetic energy of a body is twice its rest mass energy. What is the ratio of relativistic mass to rest mass of the body ? (CSE 2008)
 (a) 1 (b) 2 (c) 3 (d) 0.5
31. The length of a rod is measured in different frames of reference in which the rod moves with velocities having following values.
 1. zero 2. 60% of velocity of light 3. 2.4×10^8 m/s.
 What is the correct order of decreasing measured length ? (CSE 2007)
 (a) $1 \rightarrow 3 \rightarrow 2$ (b) $3 \rightarrow 2 \rightarrow 1$ (c) $1 \rightarrow 2 \rightarrow 3$ (d) $2 \rightarrow 1 \rightarrow 3$
32. Reference frame S' moves with velocity v relative to frame S . A particle has velocity $\vec{u}'$ relative to reference frame S' and velocity $\vec{u}$ relative to frame S . Which one of the following relations is correct taking into account the effect of relativity ? (CSE 2007)
 (a) $u = u' + v$ (b) $u' = \frac{u + v}{1 + \left(\frac{uv}{c^2}\right)}$ (c) $u = \frac{u' + v}{1 + \frac{u'v}{c^2}}$ (d) $u' = \frac{u - v}{1 + \frac{uv}{c^2}}$

33. A particle of rest mass m_0 is moving with relativistic velocity v . Then
1. Kinetic energy of particle is $\frac{1}{2} m_0 v^2$
 2. Its total energy is $\frac{m_0 c^2}{\sqrt{1 - \frac{v^2}{c^2}}}$
 3. Its momentum is $\frac{m_0 v}{\sqrt{1 - \frac{v^2}{c^2}}}$ (CSE 2007)

Select the correct options using the following code

- (a) 1, 2 and 3 (b) 1 and 2 only (c) 2 and 3 only (d) 1 and 3 only.
34. A young girl goes to a distant star and returns to earth on a rocket. What is the apparent age difference between her and her twin sister which is staying on the earth. If the distance between the earth and distant star is 30 light years and the velocity of rocket is $v = 0.8c$ (when c is the speed of light)? (CSE 2006)
- (a) 30 years (b) 45 years (c) 60 years (d) 75 years.
35. Reference frame A is moving with a velocity v_A with respect to earth and reference frame B is moving with velocity v_B with respect to reference frame A. If M_A and M_B are the masses of a body in A and B respectively and C is the speed of light, then what is the value of $\frac{M_A}{M_B}$?

- (a) $\sqrt{1 - \frac{v_B^2}{c^2}}$ (b) $\sqrt{1 - \frac{(v_A - v_B)^2}{c^2}}$
- (c) $\left[1 - \frac{(v_A - v_B)^2}{c^2}\right]^{-1/2}$ (d) $\left(1 - \frac{v_B^2}{c^2}\right)^{-1/2}$ (CSE 2005)

36. A particle of rest mass $\frac{10}{3} m$ decays at rest into two mesons of mass m in each. What is velocity of each meson in terms of velocity of light c ? (CSE 2005)
- (a) $\frac{3c}{10}$ (b) $\frac{3c}{5}$ (c) $\frac{2c}{5}$ (d) $\frac{4c}{5}$
37. Which one of the following represents relativistic version of Newton's second law of motion with rest mass m_0 and speed v ?

- (a) $F = m_0 \frac{dv}{dt} \sqrt{1 - \frac{v^2}{c^2}}$ (b) $F = m_0 \frac{dv}{dt} \left(1 - \frac{v^2}{c^2}\right)^{3/2}$
- (c) $F = m_0 \frac{dv}{dt} \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$ (d) $F = m_0 \frac{dv}{dt} \left(1 - \frac{v^2}{c^2}\right)^{-3/2}$

38. Two inertial frames A and B are moving with a relative speed of $\frac{c\sqrt{7}}{4}$. If a rod of length L is placed at an angle of 45° with respect to the direction of motion in the frame A, it will seem to have a length in the other frame B, equal to
- (a) $0.95 L$ (b) $0.88 L$ (c) $0.75 L$ (d) $0.68 L$ (CSE 2004)

39. A particle of mass m at rest decays into a particle of mass $\frac{m}{2}$ and c photon. The energy of photon is (c = speed of light in vacuum)

- (a) $\frac{3}{8} mc^2$ (b) $\frac{3}{2} mc^2$ (c) $\frac{1}{4} mc^2$ (d) $\frac{1}{2} mc^2$

40. A spaceship is approaching a source of light with a speed $0.5c$ (where c is speed of light). Light coming from the source of light as seen by a person in the spaceship travels with speed equal to

- (a) $0.5 c$ (b) c (c) $1.5 c$ (d) $2.0 c$ (CSE 2004)

41. A photographic plate placed in a magnetic field shows a track of cosmic ray particle that its energy is 870 MeV and momentum is nearly 720 MeV/ c . Then the rest mass of particle is
- (a) 1600 MeV/ c^2 (b) 850 MeV/ c^2 (c) 490 MeV/ c^2 (d) 270 MeV/ c^2

(CSE 2004)

42. At what velocity will the mass of an object become twice its rest mass ?
 (a) $\frac{c}{2}$ (b) $\frac{\sqrt{3}}{2}$ (c) $\frac{c}{\sqrt{3}}$ (d) $\frac{c}{\sqrt{2}}$
43. Minkowski-space is
 (a) two dimensional (b) three dimensional
 (c) four dimensional (d) 3n-dimensional.
44. In Minkowski space, the invariant quantity is
 (a) $dx^2 + dy^2 + dz^2$ (b) $dx^2 + dy^2 + dz^2 - c^2 dt^2$
 (c) $dx^2 + dy^2 + dz^2 + c^2 t^2$ (d) $\frac{dx^2 + dy^2 + dz^2}{c^2 t^2}$
45. The interval between two events is time like if s_{12} given by $s_{12} = [c^2(t_2 - t_1)^2 - \{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\}]^{1/2}$, is
 (a) real (b) imaginary (c) zero (d) infinity.
46. The interval between two events is space like if s_{12} given by $s_{12} = [c^2(t_2 - t_1)^2 - (x_2 - x_1)^2 - (y_2 - y_1)^2 - (z_2 - z_1)^2]^{1/2}$ is
 (a) real (b) imaginary (c) zero (d) infinity.
47. Which is invariant under Lorentz transformation ?
 (a) line element
 (b) area element
 (c) three dimensional volume element $dx dy dz$
 (d) four dimensional volume element $dx dy dz dt$.
48. In momentum four vector, the fourth component is $[E$ is energy]
 (a) ict (b) iE (c) $\frac{iE}{c}$ (d) icE .
49. Momentum four vector
 (a) is space like (b) is time like
 (c) has magnitude unity (d) has magnitude zero.
50. If λ is the actual wavelength of light emitted by a receding star, then observed wavelength λ' is given by $\lambda =$
 (a) $\frac{\lambda}{\sqrt{1 - \frac{v^2}{c^2}}}$ (b) $\lambda \sqrt{1 - \frac{v^2}{c^2}}$ (c) $\lambda \sqrt{\frac{(1 - \frac{v}{c})}{1 + \frac{v}{c}}}$ (d) $\lambda \sqrt{\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}}$
33. A photon has velocity c in an inertial frame. S moving with velocity v , then its velocity as observed from system S' will be
 (a) $c + v$
 (b) $c - v$
 (c) c
 (d) $c + v$ if both velocities are along same direction and $c - v$ if they are in opposite direction.

ANSWERS

1. (c), 2. (b), 3. (d), 4. (b), 5. (d), 6. (a), 7. (d), 8. (c), 9. (c), 10. (b),
 11. (b), 12. (b), 13. (c), 14. (c), 15. (a), 16. (b), 17. (b), 18. (d), 19. (b), 20. (a),
 21. (d), 22. (b), 23. (a), 24. (c), 25. (a), 26. (c), 27. (b), 28. (a), 29. (a), 30. (c),
 31. (c), 32. (c), 33. (c), 34. (b), 35. (a), 36. (d), 37. (d), 38. (b), 39. (d), 40. (b),
 41. (b), 42. (b), 43. (c), 44. (b), 45. (a), 46. (b), 47. (d), 48. (c), 49. (b), 50. (d),
 51. (c).

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